

Research Article

Using Homo-Separation of Variables for Solving Systems of Nonlinear Fractional Partial Differential Equations

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A new method proposed and coined by the authors as the homo-separation of variables method is utilized to solve systems of linear and nonlinear fractional partial differential equations (FPDEs). The new method is a combination of two well-established mathematical methods, namely, the homotopy perturbation method (HPM) and the separation of variables method. When compared to existing analytical and numerical methods, the method resulting from our approach shows that it is capable of simplifying the target problem at hand and reducing the computational load that is required to solve it, considerably. The efficiency and usefulness of this new general-purpose method is verified by several examples, where different systems of linear and nonlinear FPDEs are solved.

1. Introduction

In the recent years, it has turned out that many phenomena in various technical and scientific fields can be described very successfully by using fractional calculus. In particular, fractional calculus can be employed to solve many problems within the biomedical research field and get better results. Such a practical application of fractional order models is to use these models to improve the behavior and efficiency of bioelectrodes. The importance of this application is based on the fact that bioelectrodes are usually needed to be used for all types of biopotential recording and signal measurement purposes such as electroencephalography (EEG), electrocardiography (ECG), and electromyography (EMG) [1–3]. Another promising biomedical application field is proposed by Arafa et al. [4] where a fractional-order model of HIV-1 infection of CD4⁺ T cells is introduced. Other examples of applications of fractional calculus in life sciences and technology can be found in botanics [5], biology [6], rheology [7], and elastography [8].

Numerous analytical methods have been presented in the literature to solve FPDEs, such as the fractional Greens function method [9], the Fourier transform method [2],

the Sumudu transform method [10], the Laplace transform method, and the Mellin transform method [11]. Some numerical methods have also widely been used to solve systems of FPDEs, such as the variational iteration method [12], the Adomian decomposition method [2], the homotopy perturbation method [13] and the homotopy analysis method [14]. Some of these methods use specific transformations and others give the solution as a series which converges to the exact solution.

In addition, some numerical methods use a combination of utilizing specific transformations and obtaining series which converge to the exact solutions. An example of such a method is the iterative Laplace transform method which is a combination of the Laplace transform method and an iterative method [15]. Another such a combination is the homotopy perturbation transformation method, which is constructed by combining two powerful methods, namely, the Laplace transform method and the homotopy perturbation method [16]. A third example is the Sumudu decomposition method, which is a combination of the Sumudu transform method and Adomian decomposition method [17]. A fourth such an approach is combining the Sumudu transformation method with the homotopy perturbation

method, which gives a new method called the homotopy perturbation Sumudu transform method [18].

Recently, the homotopy perturbation method and the Adomian decomposition method are frequently used for solving nonlinear FPDE problems.

According to the combinational methods mentioned above, it is possible to notice that HPM has a strong potential to be combined with another method to produce a more efficient approach. The main reason is that HPM is an efficient method for solving PDEs and ODEs (ordinary differential equations) with integer or fractional order. Recently, Karbalaie et al. [19] found the exact solution of one-dimensional FPDEs by using truncated versions of modified HPM. Therefore, by getting inspiration of the ideas, methods, and tools of previous works, the novel approach called the homo-separation of variables method is developed and utilized to find the exact solutions of systems of FPDEs. The methods presented in Karbalaie et al. [19] are extended and developed further to achieve methods that can solve n-dimensional systems of FPDEs. This new approach is constructed by a smart combination of HPM and the separation of variables method. By using this method, the system of FPDEs to be solved is changed into a system of FODEs. Consequently, the original problem is simplified considerably which makes it more straightforward and easier to solve.

The structure of the current paper is as follows. For easy reader-friendly reference, some basic definitions and properties of fractional differential equations are presented in Section 2. In Section 3, the homo-separation of variables method is described. After that, in Section 4, three examples are presented to show the simplicity and efficiency (at the same time) of the homo-separation of variables method. Finally, in Section 5, relevant conclusions are drawn.

2. Basic Definitions and Properties

In this section, to make it easier to introduce the new approach of this research work, a number of good-to-know concepts and properties within the field of fractional differential equations are defined and presented briefly.

Definition 1. A real function $f(t)$, $t > 0$, is said to be in the space C_σ , $\sigma \in \mathbb{R}$, if there exists a real number $p > \sigma$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0, \infty)$, and it is said to be in the space C_σ^m if $f^m \in C_\sigma$, $m \in \mathbb{N}$.

Definition 2. The left sided Riemann-Liouville fractional integral of order $\alpha \geq 0$, of a function $f \in C_\sigma$, $\sigma \geq -1$, is defined as

$$J_t^\alpha f(t) \stackrel{\text{def}}{=} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad (1)$$

where $\alpha > 0$, $t > 0$, and $\Gamma(\alpha) = \int_0^\infty r^{\alpha-1} e^{-r} dr$ is the gamma function.

Properties of the operator J_t^α (for $f \in C_\mu$, $\mu \geq -1$, $\alpha, \beta \geq 0$, $\gamma \geq -1$) can be found in [20] and are defined as follows:

$$(1) J_t^0 f(t) = f(t),$$

$$(2) J_t^\alpha J_t^\beta f(t) = J_t^{\alpha+\beta} f(t),$$

$$(3) J_t^\alpha t^\gamma = (\Gamma(\gamma+1)/\Gamma(\alpha+\gamma+1))t^{\alpha+\gamma}.$$

Definition 3. The left sided Caputo fractional derivative of f , ($f \in C_\mu^n$, $n \in \mathbb{N} \cup \{0\}$), in Caputo sense is defined by [11] as follows:

$$D_t^\alpha f(t) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, & n-1 < \alpha \leq n, \\ D_t^n f(t), & \alpha = n. \end{cases} \quad (2)$$

Note that, according to [21], (1) and (2) become

$$J_t^\alpha f(x, t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(x, \tau) d\tau, \quad \alpha > 0, t > 0,$$

$$D_t^\alpha f(x, t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(x, \tau) d\tau, \quad n-1 < \alpha \leq n. \quad (3)$$

Definition 4. The single-parameter and the two-parameter variants of the Mittag-leffler function are denoted by $E_\alpha(t)$ and $E_{\alpha,\beta}(t)$, respectively. These two variants are relevant to be considered here because of their connection with fractional calculus and are defined as

$$E_\alpha(t) = \sum_{j=0}^\infty \frac{t^j}{\Gamma(\alpha j + 1)}, \quad \alpha > 0, t \in \mathbb{C}, \quad (4)$$

$$E_{\alpha,\beta}(t) = \sum_{j=0}^\infty \frac{t^j}{\Gamma(\alpha j + \beta)}, \quad \alpha, \beta > 0, t \in \mathbb{C}.$$

The k th derivatives of these two variants are

$$E_\alpha^{(k)}(t) = \frac{d^k}{dt^k} E_\alpha(t) = \sum_{j=0}^\infty \frac{(k+j)!t^j}{j!\Gamma(\alpha j + \alpha k + 1)}, \quad k = 0, 1, 2, \dots, \quad (5)$$

$$E_{\alpha,\beta}^{(k)}(t) = \frac{d^k}{dt^k} E_{\alpha,\beta}(t) = \sum_{j=0}^\infty \frac{(k+j)!t^j}{j!\Gamma(\alpha j + \alpha k + \beta)}, \quad k = 0, 1, 2, \dots$$

Some special cases and properties of the Mittag-Leffler functions are the following:

$$(1) E_{\alpha,1}(t) = E_\alpha(t),$$

$$(2) E_{1,1}(t) = E_1(t) = e^t,$$

$$(3) E_{2,1}(t^2) = \cosh(t),$$

$$(4) E_{2,2}(t^2) = \sinh(t)/t,$$

- (5) $E_{\alpha,\beta}(t) + E_{\alpha,\beta}(-t) = 2E_{2\alpha,\beta}(t^2),$
- (6) $E_{\alpha,\beta}(t) - E_{\alpha,\beta}(-t) = 2tE_{2\alpha,\alpha+\beta}(t^2),$
- (7) $D_t^\alpha(t^{\alpha-1}E_{\alpha,\alpha}(at^\alpha)) = t^{\alpha-1}E_{\alpha,\alpha}(at^\alpha).$

Further properties of the Mittag-Leffler functions can be found in [21]. These functions are generalizations of the exponential function; therefore, most linear differential equations of fractional order (FPDEs) have solutions that are expressed in terms of these functions.

Important Note. According to [22], $E_\alpha(at^\alpha)$ cannot satisfy the following semigroup property:

$$E_\alpha(a(t+s)^\alpha) = E_\alpha(at^\alpha)E_\alpha(as^\alpha), \quad t, s \geq 0, \quad a \in \mathbb{R} \quad (6)$$

for any noninteger $\alpha > 0$. The above property is not true unless in the special cases where $\alpha = 1$ or 0 . Unfortunately, this misunderstanding can be found in a number of publications, which can be misleading and confusing.

Definition 5. Consider the following relaxation-oscillation equation, which is discussed in [23]:

$$D_t^\alpha u(t) + Au(t) = f(t), \quad t > 0, \quad (7)$$

subject to the initial condition

$$\left. \frac{\partial^i}{\partial t^i} u(t) \right|_{t=0} = u^i(0) = b_i \quad i = 1, 2, \dots, n-1, \quad (8)$$

where b_i ($i = 1, 2, \dots, n-1$) are real constants and $n-1 < \alpha \leq n$. The solution of (7) is obtained according to [23] as follows:

$$u(t) = \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-A(t-\tau)^\alpha) f(\tau) d\tau + \sum_{j=0}^{n-1} b_j D_t^{\alpha-j-1} (t^{\alpha-1} E_{\alpha,\alpha}(-At^\alpha)), \quad (9)$$

where $E_{\alpha,\alpha}$ is the Mittag-Leffler function. It is easy to realize that if $0 < \alpha \leq 1$, then the solution of (7) becomes as follows:

$$u(t) = \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-A(t-\tau)^\alpha) f(\tau) d\tau + D_t^{\alpha-1} (t^{\alpha-1} E_{\alpha,\alpha}(-At^\alpha)), \quad (10)$$

which will be used to solve the systems of FPDEs in the examples discussed in this paper.

Definition 6. According to [24], consider the following linear system of fractional differential equations with its initial condition:

$$D_t^\alpha \mathbf{x}(t) = A\mathbf{x}(t), \quad 0 < t \leq a, \quad \mathbf{x}(0) = \mathbf{x}_0, \quad (11)$$

where $\mathbf{x} \in \mathbb{R}^n$, $a > 0$, the matrix $A \in \mathbb{R}^n \times \mathbb{R}^n$ and D_t^α is the Caputo fractional derivative of order α , where $0 < \alpha \leq 1$. The general analytical solution for (11) is given by

$$\mathbf{x}(t) = c_1 \mathbf{u}^{(1)} E_\alpha(\lambda_1 t^\alpha) + c_2 \mathbf{u}^{(2)} E_\alpha(\lambda_2 t^\alpha) + \dots + c_n \mathbf{u}^{(n)} E_\alpha(\lambda_n t^\alpha), \quad (12)$$

where c_1, c_2, \dots, c_n are arbitrary constants, E_α is the Mittag-Leffler function, and $\lambda_1, \lambda_2, \dots, \lambda_n$ and $\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \dots, \mathbf{u}^{(n)}$ are the eigenvalues and the corresponding eigenvectors of the characteristic equation $(A - \lambda_i \mathbf{I}) \mathbf{u}^{(i)} = 0$, where $i = 1, 2, \dots, n$ and \mathbf{I} is an $n \times n$ identity matrix.

3. The Homo-Separation of Variables Method

This approach is achieved by a novel combination of HPM and separation of variables. In this section, the algorithm of this new method is presented briefly by considering the following system of FPDEs:

$$D_t^{\alpha_i} u_i = f_i(\bar{x}, t) + \mathcal{L}_i(u_1, u_2, \dots, u_n) + \mathcal{N}_i(u_1, u_2, \dots, u_n), \quad i = 1, 2, \dots, n, \quad (13)$$

subject to the initial conditions

$$u_i(\bar{x}, 0) = g_i(\bar{x}), \quad (14)$$

where $\bar{x} \in \mathbb{R}^{n-1}$, and for $i = 1, 2, \dots, n$, the terms \mathcal{L}_i represent linear operators, the terms \mathcal{N}_i are nonlinear operators, while the terms f_i represent known analytical functions. In addition, $D_t^{\alpha_i} = \partial^{\alpha_i} / \partial t^{\alpha_i}$ is the Caputo fractional derivative of order α_i , where $0 < \alpha_i \leq 1$ for $i = 1, 2, \dots, n$.

According to the homotopy perturbation technique (HPM), we can construct the following homotopies:

$$(p-1)(D_t^{\alpha_i} V_i - D_t^{\alpha_i} u_{i0}) + p(D_t^{\alpha_i} V_i - \widetilde{\mathcal{L}}_i - \widetilde{\mathcal{N}}_i - f_i(\bar{x}, t)) = 0, \quad i = 1, 2, \dots, n, \quad (15)$$

where $\widetilde{\mathcal{L}}_i = \mathcal{L}_i(V_1, V_2, \dots, V_n)$, $\widetilde{\mathcal{N}}_i = \mathcal{N}_i(V_1, V_2, \dots, V_n)$, and the homotopy parameter, denoted by p , is considered as small ($p \in [0, 1]$). In addition, for $i = 1, 2, \dots, n$, $u_{i0} = u_{i0}(\bar{x}, t)$ is an initial approximation of the solution of (13) which satisfies the initial condition in (14).

We can assume that the solution of (15) can be expressed as a power series in p , as given below in (16).

$$V_1 = \sum_{j=0}^{\infty} p^j V_{1j} = V_{10} + pV_{11} + p^2V_{12} + p^3V_{13} + \dots, \\ V_2 = \sum_{j=0}^{\infty} p^j V_{2j} = V_{20} + pV_{21} + p^2V_{22} + p^3V_{23} + \dots, \\ \vdots \\ V_n = \sum_{j=0}^{\infty} p^j V_{nj} = V_{n0} + pV_{n1} + p^2V_{n2} + p^3V_{n3} + \dots. \quad (16)$$

By substituting (16) into (15) and equating the terms with identical powers of p , a set of equations is obtained as follows:

$$p^0 : D_t^{\alpha_i} V_{i0} - D_t^{\alpha_i} u_{i0} = 0, \tag{17}$$

$$p^1 : D_t^{\alpha_i} V_{i1} + D_t^{\alpha_i} u_{i0} \tag{18}$$

$$-\mathcal{L}_i(V_{10}, V_{20}, \dots, V_{n0}) - \mathcal{M}_{i0} - f_i = 0, \tag{19}$$

$$\vdots$$

$$p^j : D_t^{\alpha_i} V_{ij} - \mathcal{L}_i(V_{1j-1}, V_{2j-1}, \dots, V_{nj-1}) - \mathcal{M}_{ij-1} = 0,$$

where the terms \mathcal{M}_{ij} ($i = 1, 2, \dots, n$ and $j = 0, 1, \dots$) are obtained from (17)–(19) by equating the coefficients of the nonlinear operators \mathcal{N}_{ij} ($i = 1, 2, \dots, n$ and $j = 0, 1, \dots$) with the identical powers j of p , that is, the terms containing p^j .

In case the p -parameter is considered as small, the best approximate solution of (13) can be readily obtained as follows:

$$u_i(\bar{x}, t) = \lim_{p \rightarrow 1} \sum_{j=0}^{\infty} p^j V_{ij}(\bar{x}, t) \tag{20}$$

$$= V_{i0}(\bar{x}, t) + V_{i1}(\bar{x}, t) + V_{i2}(\bar{x}, t) + \dots,$$

$$i = 1, 2, \dots, n,$$

where $\bar{x} = (x_1, x_2, \dots, x_{n-1})$.

If in (20) there exists some $V_{iN}(\bar{x}, t) = 0$, where $N \geq 1$ and $i = 1, 2, \dots, n$, then the exact solution can be written in the following form:

$$u_i(\bar{x}, t) = V_{i0}(\bar{x}, t) + pV_{i1}(\bar{x}, t) + \dots + p^{N-1}V_{iN-1}(\bar{x}, t), \tag{21}$$

$$i = 1, 2, \dots, n.$$

For simplicity, we assume that $V_{i1}(\bar{x}, t) \equiv 0$ in (21), which means that the exact solution in (13) is simplified as follows:

$$u_i(\bar{x}, t) = V_{i0}(\bar{x}, t), \quad i = 1, 2, \dots, n. \tag{22}$$

Now when solving (17), we obtain the following result:

$$V_{i0}(\bar{x}, t) = u_{i0}(\bar{x}, t), \quad i = 1, 2, \dots, n. \tag{23}$$

By using (22) and (23) we have

$$u_i(\bar{x}, t) = V_{i0}(\bar{x}, t) = u_{i0}(\bar{x}, t), \quad i = 1, 2, \dots, n. \tag{24}$$

The core of the new method proposed in this research work is to formulate the initial approximation of (13) in the form of separation of variables, as follows:

$$u_i(\bar{x}, t) = u_{i0}(\bar{x}, t) = u_i(\bar{x}, 0) c_{i1}(t) \tag{25}$$

$$+ \left(\sum_{j=1}^n D_{x_j} u_i(\bar{x}, 0) \right) c_{i2}(t)$$

$$= g_i(\bar{x}) c_{i1}(t) + \left(\sum_{j=1}^n D_{x_j} g_i(\bar{x}) \right) c_{i2}(t), \quad i = 1, 2, \dots, n.$$

The task now is to find the terms $c_{i1}(t)$ and $c_{i2}(t)$ to obtain the exact solution of the system of FPDEs in (13). Since (25) satisfies the initial conditions in (14), we get

$$c_{i1}(0) = 1, \quad c_{i2}(0) = 0, \quad i = 1, 2, \dots, n. \tag{26}$$

By substituting (25) into (18), we get

$$D_t^{\alpha_i} V_{i1} = -D_t^{\alpha_i} \left(g_i(\bar{x}) c_{i1}(t) + \left(\sum_{j=1}^n D_{x_j} g_i(\bar{x}) \right) c_{i2}(t) \right) \tag{27}$$

$$-L_i - M_{i0} + f_i(\bar{x}, t) \equiv 0,$$

where $i = 1, 2, \dots, n$ and

$$L_i = \mathcal{L}_i \left(g_1(\bar{x}) c_{11}(t) + \left(\sum_{j=1}^n D_{x_j} g_1(\bar{x}) \right) c_{12}(t), \dots, \right. \tag{28}$$

$$\left. \left(g_n(\bar{x}) c_{n1}(t) + \left(\sum_{j=1}^n D_{x_j} g_n(\bar{x}) \right) c_{n2}(t) \right) \right),$$

$$M_{i0} = \mathcal{M}_{i0} \left(g_1(\bar{x}) c_{11}(t) + \left(\sum_{j=1}^n D_{x_j} g_1(\bar{x}) \right) c_{12}(t), \dots, \right. \tag{28}$$

$$\left. \left(g_n(\bar{x}) c_{n1}(t) + \left(\sum_{j=1}^n D_{x_j} g_n(\bar{x}) \right) c_{n2}(t) \right) \right).$$

Obviously, our goal is finally achieved and the system of FPDEs is changed into a system of FODEs. Consequently, the problem at hand is simplified considerably as it is well known that solving a system of FODEs is generally more straightforward and much easier than solving the corresponding system of FPDEs.

Finally, the target unknowns $c_{i1}(t)$ and $c_{i2}(t)$ can be obtained by utilizing and solving the system of FODEs and the initial conditions in (26).

4. Applications

In this section, we illustrate the applicability, simplicity, and efficiency of the homo-separation of variables method for solving systems of linear and nonlinear FPDEs.

Example 7. Consider the following system of linear FPDEs:

$$D_t^\alpha u = v_x - v - u, \tag{29}$$

$$D_t^\alpha v = u_x - v - u,$$

where $0 < \alpha \leq 1$, subject to the initial conditions

$$u(x, 0) = \sinh(x), \quad v(x, 0) = \cosh(x). \tag{30}$$

To solve (29) and (30) by using the proposed homo-separation of variables method, we choose the initial approximations for (29) as follows:

$$u_0(x, t) = u(x, 0) b_1(t) + D_x u(x, 0) b_2(t), \tag{31}$$

$$v_0(x, t) = v(x, 0) c_1(t) + D_x v(x, 0) c_2(t).$$

By substituting the two initial conditions given by (30) into these two initial approximations, we get the following initial approximations:

$$\begin{aligned} u_0(x, t) &= b_1(t) \sinh(x) + b_2(t) \cosh(x), \\ v_0(x, t) &= c_1(t) \cosh(x) + c_2(t) \sinh(x). \end{aligned} \tag{32}$$

By considering and following the approach which resulted in (27) and substituting the results presented by (32) into the system of linear FPDEs in (29), we get

$$\begin{aligned} D_t^\alpha u_1 &= \sinh(x) (D_t^\alpha b_1(t) + b_1(t) - c_1(t) + c_2(t)) \\ &\quad + \cosh(x) (D_t^\alpha b_2(t) + b_2(t) + c_1(t) - c_2(t)) \equiv 0, \\ D_t^\alpha v_1 &= \cosh(x) (D_t^\alpha c_1(t) + c_1(t) - b_1(t) + b_2(t)) \\ &\quad + \sinh(x) (D_t^\alpha c_2(t) + c_2(t) + b_1(t) - b_2(t)) \equiv 0. \end{aligned} \tag{33}$$

By solving the latter system of linear FPDEs presented in (33), we obtain the following system of linear FODEs:

$$\begin{aligned} D_t^\alpha b_1(t) + b_1(t) - c_1(t) + c_2(t) &= 0, \\ D_t^\alpha b_2(t) + b_2(t) + c_1(t) - c_2(t) &= 0, \\ D_t^\alpha c_1(t) + c_1(t) - b_1(t) + b_2(t) &= 0, \\ D_t^\alpha c_2(t) + c_2(t) + b_1(t) - b_2(t) &= 0 \end{aligned} \tag{34}$$

together with the initial conditions

$$\begin{aligned} b_2(0) &= 1, & b_1(0) &= 0, \\ c_1(0) &= 1, & c_2(0) &= 0. \end{aligned} \tag{35}$$

By solving (34) and (35) by considering and applying Definition 6 which is discussed in Section 2, we obtain the following results:

$$\begin{aligned} b_1(t) &= \frac{1}{2} (E_\alpha(t^\alpha) + E_\alpha(-t^\alpha)), \\ b_2(t) &= \frac{1}{2} (-E_\alpha(t^\alpha) + E_\alpha(-t^\alpha)), \\ c_1(t) &= \frac{1}{2} (E_\alpha(t^\alpha) + E_\alpha(-t^\alpha)), \\ c_2(t) &= \frac{1}{2} (-E_\alpha(t^\alpha) + E_\alpha(-t^\alpha)). \end{aligned} \tag{36}$$

By substituting (36) into (32) and applying Definition 4, in particular properties (2), (5) and (6) which are discussed in Section 2, we obtain the exact solution for the system of linear FPDEs in (29), as follows:

$$\begin{aligned} u(x, t) &= E_{2\alpha, 1}(t^{2\alpha}) \sinh(x) - t^\alpha E_{2\alpha, \alpha+1}(t^{2\alpha}) \cosh(x), \\ v(x, t) &= E_{2\alpha, 1}(t^{2\alpha}) \cosh(x) - t^\alpha E_{2\alpha, \alpha+1}(t^{2\alpha}) \sinh(x). \end{aligned} \tag{37}$$

If we now put $\alpha \rightarrow 1$ in (37) or solve (29) and (30) for $\alpha = 1$, we obtain the following exact solution for the corresponding system of linear PDEs:

$$\begin{aligned} u(x, t) &= \sinh(x - t), \\ v(x, t) &= \cosh(x - t). \end{aligned} \tag{38}$$

Example 8. Consider the following system of inhomogeneous FPDEs:

$$\begin{aligned} D_t^\alpha u + u_x v + u &= 1, \\ D_t^\beta v - v_x u - v &= 1, \end{aligned} \tag{39}$$

where $0 < \alpha$ and $\beta \leq 1$, subject to the initial conditions

$$u(x, 0) = e^x, \quad v(x, 0) = e^{-x}. \tag{40}$$

To solve (39) and (40) by using the new homo-separation of variables method, we choose the initial approximations for (39) as follows:

$$\begin{aligned} u_0(x, t) &= u(x, 0) c_1(t) + D_x u(x, 0) c_2(t), \\ v_0(x, t) &= v(x, 0) b_1(t) + D_x v(x, 0) b_2(t). \end{aligned} \tag{41}$$

By substituting the two initial conditions given by (40) into these two initial approximations, we get the following initial approximations:

$$\begin{aligned} u_0(x, t) &= e^x c_1(t) + e^x c_2(t), \\ v_0(x, t) &= e^{-x} b_1(t) - e^{-x} b_2(t). \end{aligned} \tag{42}$$

By considering and following the approach which resulted in (27) and substituting the results presented by (42) into the system of inhomogeneous FPDEs that we would like to solve, which is presented by (39), we get

$$\begin{aligned} D_t^\alpha u_1 &= e^x (D_t^\alpha (c_1(t) + c_2(t)) + (c_1(t) + c_2(t))) \\ &\quad + (c_1(t) + c_2(t)) (b_1(t) - b_2(t)) - 1 \equiv 0, \\ D_t^\beta v_1 &= e^{-x} (D_t^\beta (b_1(t) - b_2(t)) - (b_1(t) - b_2(t))) \\ &\quad + (c_1(t) + c_2(t)) (b_1(t) - b_2(t)) - 1 \equiv 0. \end{aligned} \tag{43}$$

By solving the latter system of FPDEs presented in (43), we obtain the following system of FODEs:

$$\begin{aligned} D_t^\alpha (c_1(t) + c_2(t)) + (c_1(t) + c_2(t)) &= 0, \\ D_t^\beta (b_1(t) - b_2(t)) - (b_1(t) - b_2(t)) &= 0, \\ (c_1(t) + c_2(t)) (b_1(t) - b_2(t)) - 1 &= 0 \end{aligned} \tag{44}$$

together with the initial conditions

$$\begin{aligned} b_1(0) &= 1, & b_2(0) &= 0, \\ c_1(0) &= 1, & c_2(0) &= 0. \end{aligned} \tag{45}$$

By solving (44) and (45) by considering and applying Definition 5 which is discussed in Section 2, we obtain the following results:

$$\begin{aligned} c_1(t) + c_2(t) &= E_\alpha(-t^\alpha), \\ b_1(t) - b_2(t) &= E_\beta(t^\beta), \\ E_\alpha(-t^\alpha) E_\beta(t^\beta) &= 1. \end{aligned} \tag{46}$$

By substituting (46) into (42), we obtain the exact solution for the system of inhomogeneous FPDEs presented in (39), as follows:

$$\begin{aligned} u(x, t) &= e^x E_\alpha(-t^\alpha), \\ v(x, t) &= e^{-x} E_\beta(t^\beta), \\ E_\alpha(-t^\alpha) E_\beta(t^\beta) &= 1. \end{aligned} \tag{47}$$

Thus, this means that $u(x, t) = 1/v(x, t)$. If we now put $\alpha, \beta \rightarrow 1$ in (47), or solve (39) and (40) for $\alpha = \beta = 1$, we obtain the following exact solution for the corresponding system of inhomogeneous PDEs:

$$\begin{aligned} u(x, t) &= e^{x-t}, \\ v(x, t) &= e^{-x+t}. \end{aligned} \tag{48}$$

Example 9. Consider the following system of nonlinear FPDEs:

$$\begin{aligned} D_t^\alpha u &= -u - v_x w_y - v_y w_x, \\ D_t^\beta v &= v - u_x w_y - u_y w_x, \\ D_t^\gamma w &= w - u_x v_y - u_y v_x, \end{aligned} \tag{49}$$

where $0 < \alpha$ and $\beta, \gamma \leq 1$, subject to the initial conditions

$$\begin{aligned} u(x, y, 0) &= e^{x+y}, & v(x, y, 0) &= e^{x-y}, \\ w(x, y, 0) &= e^{-x+y}. \end{aligned} \tag{50}$$

To solve (49) and (50) by using the new homo-separation of variables method, we choose the initial approximations for (49) as follows:

$$\begin{aligned} u_0(x, y, t) &= u(x, y, 0) b_1(t) \\ &\quad + (D_x u(x, y, 0) + D_y u(x, y, 0)) b_2(t), \\ v_0(x, y, t) &= v(x, y, 0) c_1(t) \\ &\quad + (D_x v(x, y, 0) + D_y v(x, y, 0)) c_2(t), \\ w_0(x, y, t) &= w(x, y, 0) d_1(t) \\ &\quad + (D_x w(x, y, 0) + D_y w(x, y, 0)) d_2(t). \end{aligned} \tag{51}$$

By substituting the initial conditions given by (50) into these initial approximations, we get the following initial approximations:

$$\begin{aligned} u_0(x, y, t) &= e^{x+y} b_1(t) + 2e^{x+y} b_2(t), \\ v_0(x, y, t) &= e^{x-y} c_1(t), \\ w_0(x, y, t) &= e^{-x+y} d_1(t). \end{aligned} \tag{52}$$

By considering and following the approach which resulted in (27) and substituting the results presented by (52) into the system of nonlinear FPDEs that we would like to solve, which is presented by (49), we get

$$\begin{aligned} D_t^\alpha u_1 &= e^{-x+y} (D_t^\alpha b_1(t) + 2D_t^\alpha b_2(t) + b_1(t) + 2b_2(t)) \equiv 0, \\ D_t^\beta D_t v_1 &= e^{x-y} (D_t^\beta c_1(t) - c_1(t)) \equiv 0, \\ D_t^\gamma w_1 &= e^{-x+y} (D_t^\gamma d_1(t) - d_1(t)) \equiv 0. \end{aligned} \tag{53}$$

By solving the latter system of FPDEs presented in (53), we obtain the following system of FODEs:

$$\begin{aligned} D_t^\alpha (b_1(t) + 2b_2(t)) + (b_1(t) + 2b_2(t)) &= 0, \\ D_t^\beta c_1(t) - c_1(t) &= 0, \end{aligned} \tag{54}$$

$$D_t^\gamma d_1(t) - d_1(t) = 0,$$

together with the initial conditions

$$\begin{aligned} b_1(0) &= 1, & b_2(0) &= 0, \\ c_1(0) &= 1, & c_2(0) &= 0, \\ d_1(0) &= 1, & d_2(0) &= 0. \end{aligned} \tag{55}$$

By solving (54) and (55) by considering and applying Definition 5 which is discussed in Section 2 of this paper, we obtain the following results:

$$\begin{aligned} b_1(t) - 2b_2(t) &= E_\alpha(-t^\alpha), \\ c_1(t) &= E_\beta(t^\beta), \\ d_1(t) &= E_\gamma(t^\gamma). \end{aligned} \tag{56}$$

By substituting (56) into (52), we obtain the exact solution for the system of nonlinear FPDEs in (49), as follows:

$$\begin{aligned} u(x, y, t) &= e^{x+y} E_\alpha(-t^\alpha), \\ v(x, y, t) &= e^{x-y} E_\beta(t^\beta), \\ w(x, y, t) &= e^{-x+y} E_\gamma(t^\gamma). \end{aligned} \tag{57}$$

If we now put $\alpha, \beta, \gamma \rightarrow 1$ in (57), or solve (49) and (50) for $\alpha = \beta = \gamma = 1$, we obtain the following exact solution for the corresponding system of nonlinear PDEs:

$$\begin{aligned} u(x, y, t) &= e^{x+y-t}, \\ v(x, y, t) &= e^{x-y+t}, \\ w(x, y, t) &= e^{-x+y+t}. \end{aligned} \tag{58}$$

5. Conclusions

In this paper, a novel approach was introduced and utilized to solve linear and nonlinear systems of FPDEs. The new technique was coined by the authors of this paper as the homo-separation of variables method. In this research work, it was demonstrated through different examples how this new method can be used for solving various systems of FPDEs.

When compared with the existing published methods, it is easy to notice that the new method has many advantages. It is straightforward, easy to understand, and fast, requiring much less computations to perform a limited number of steps of the simple procedure that can be applied to find the exact solution of a wide range of types of FPDE equations' systems. Furthermore, there is no need for using linearization or restrictive assumptions when employing this new method.

The basic principle of the current method is to perform an elegant combination (which is simple and smart) of two existing techniques. These two techniques are the popular homotopy perturbation method (HPM) that was originally proposed by He [25, 26] and another popular approach called separation of variables. Finally, the resulting homo-separation of variables method, which is analytical, can be used to solve various systems of PDEs with integer and fractional order, for example, with respect to time.

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