Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2013, Article ID 684757, 8 pages
http://dx.doi.org/10.1155/2013/684757

Research Article

Optimal Consumption in a Stochastic Ramsey Model with Cobb-Douglas Production Function

Md. Azizul Baten[1] and Anton Abdulbasah Kamil[2]

[1] *Department of Decision Science, School of Quantitative Sciences, Universiti Utara Malaysia (UUM), Sintok, 06010 Kedah, Malaysia*
[2] *Mathematics Program, School of Distance Education, Universiti Sains Malaysia (USM), 11800 Penang, Malaysia*

Correspondence should be addressed to Md. Azizul Baten; baten_math@yahoo.com

Received 9 November 2012; Revised 7 January 2013; Accepted 16 January 2013

Academic Editor: Aloys Krieg

A stochastic Ramsey model is studied with the Cobb-Douglas production function maximizing the expected discounted utility of consumption. We transformed the Hamilton-Jacobi-Bellman (HJB) equation associated with the stochastic Ramsey model so as to transform the dimension of the state space by changing the variables. By the viscosity solution method, we established the existence of viscosity solution of the transformed Hamilton-Jacobi-Bellman equation associated with this model. Finally, the optimal consumption policy is derived from the optimality conditions in the HJB equation.

1. Introduction

In financial decision-making problems, Merton's [1, 2] papers seemed to be pioneering works. In his seminal work, Merton [2] showed how a stochastic differential for the labor supply determined the stochastic processes for the short-term interest rate and analyzed the effects of different uncertainties on the capital-to-labor ratio. The existence and uniqueness of solutions to the state equation of the Ramsay problem [2] is not yet available. In this study, we turned to Merton's [2] original problem that is revisited considering the growth model for the Cobb-Douglas production function in the finite horizon. Let us define the following quantities:

$\tau_k = \inf\{t \geq 0 : K_t = 0\}$,

K_t = capital stock at time $t \geq 0$,

L_t = labor supply at time $t \geq 0$,

λ = constant rate of depreciation, $\lambda \geq 0$,

$c_t K_t$ = consumption rate at time $t \geq 0, 0 \leq c_t \leq 1$,

$c_t K_t / L_t$ = totality of consumption rate per labor;

$F(K, L) = AK^{\alpha} L^{1-\alpha}$ with $0 < \alpha < 1$ and A is a constant, production function producing the commodity for the capital stock $K > 0$ and the labor supply $L > 0$,

n = rate of labor growth (nonzero constant),

σ = non-zero constant coefficients,

ρ = discount rate $\rho > 0$,

$U(c)$ = utility function for the consumption rate $c \geq 0$,

W_t = one-dimensional standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, P)$ endowed with the natural filtration $\mathcal{F}_t$ generated by $\sigma(W_s, s \leq t)$.

Let us assume that $c = \{c_t\}$ is a consumption policy per capita such that c_t is nonnegative $\mathcal{F}_t = \sigma(W_s, s \leq t)$, a progressively measurable process,

$$\int_0^t c_s ds < \infty \quad \text{a.s. } \forall t \geq 0, \tag{1}$$

and we denote by $\mathscr{A}$ the set of all consumption policies $\{c_t\}$ per capita.

The utility function $U(c)$ is assumed to have the following properties:

$$\begin{aligned} &U \in C[0,\infty)\bigcap C^2(0,\infty),\\ &U(c): \text{strictly concave on } [0,\infty),\\ &U'(c): \text{strictly decreasing}, \quad U''<0 \text{ for } c>0,\\ &U'(\infty)=U'(0+)=0, \qquad U'(0+)=U(\infty)=\infty. \end{aligned} \tag{2}$$

Following Merton [2], we make the following assumption on the Cobb-Douglas production function $F(K,L)$:

$$\begin{aligned} &F(\gamma K,\gamma L)=\gamma F(K,L), \quad \text{for } \gamma>0,\\ &F_K(K,L)>0, \qquad F_L(K,L)>0, \qquad F_{KK}(K,L)<0\\ &F_{LL}(K,L)<0,\\ &F(0,L)=F(K,0)=0, \qquad F_K(0+,L)<\infty,\\ &F_K(\infty,L)=0, \quad L>0. \end{aligned} \tag{3}$$

We are concerned with the economic growth model to maximize the expected discount utility of consumption

$$J(c)=E\left[\int_0^T e^{-\rho t}U\left(\frac{c_tK_t}{L_t}\right)dt\right] \tag{4}$$

per labor with a horizon T over the class $c\in\mathscr{A}$ subject to the capital stock K_t, and the labor supply L_t is governed by the stochastic differential equation

$$\begin{aligned} dK_t&=\left[F(K_t,L_t)-\lambda K_t-c_tK_t\right]dt \quad K_0=K,\ K>0,\\ dL_t&=nL_tdt+\sigma L_tdW_t, \quad L_0=L,\ L>0. \end{aligned} \tag{5}$$

This optimal consumption problem has been studied by Merton [2], Kamien and Schwartz [3], Koo [4], Morimoto and Kawaguchi [5], Morimoto [6], and Zeldes [7]. Recently, this kind of problem is treated by Baten and Sobhan [8] for one-sector neoclassical growth model with the constant elasticity of substitution (CES) production function in the infinite time horizon case. The studies of Ramsey-type stochastic growth models are also available in Amilon and Bermin [9], Bucci et al. [10], Posch [11], and Roche [12]; comprehensive coverage of this subject can be found, for example, in the books of Chang [13], Malliaris et al., [14], Turnovsky [15, 16], and Walde [17–19]. Continuous-time steady-state studies under lower-dimensional uncertainty carried out, for example, by Merton [2] and Smith [20] within a Ramsey-type setup, and, for example, by Bourguignon [21], Jensen and Richter [22], and Merton [2] within a Solow-Swan-type setup. But these papers did not deal with establishing the existence of viscosity solution of the transformed Hamilton-Jacobi-Bellman equation, and they did not derive the optimal consumption policy from the optimality conditions in the HJB equation associated with the stochastic Ramsey problem, which we have dealt with in this paper.

On the other hand, Oksendal [23] considered a cash flow modeled with geometric Brownian motion to maximize the expected discounted utility of consumption rate for a finite horizon with the assumption that the consumer has a logarithmic utility for his/her consumption rate. He added a jump term (represented by a Poissonian random measure) in a cash flow model. The problem discussed in Oksendal [23] is related to the optimal consumption and portfolio problems associated with a random time horizon studied in Blanchet-Scalliet et al., [24], Bouchard and Pham [25], and Blanchet-Scalliet et al., [26]. However, our paper's approach is different.

By the principle of optimality, it is natural that u solves the general (two-dimensional) Hamilton-Jacobi-Bellman (in short, HJB) equation

$$\begin{aligned} &-\rho u(K,L)+\frac{1}{2}\sigma^2L^2u_{LL}(K,L)+nLu_L(K,L)\\ &\quad+\{F(K,L)-\lambda K\}u_K(K,L)\\ &\quad+U^*(u_K(K,L),L)=0,\\ &\qquad u(0,L)=0, \quad K>0,\ L>0, \end{aligned} \tag{6}$$

where $U^*(u_K,L)=\max_{c>0}\{U(cK/L)-cKu_K\}$ and u_K, u_L, and u_{LL} are partial derivatives of $u(K,L)$ with respect to K and L.

The technical difficulty in solving the problem lies in the fact that the HJB equation (6) is a parabolic PDE with two spatial variables K and L. We apply the viscosity method of Fleming and Soner [27] and Soner [28] to this problem to show that the transformed one-dimensional HJB equation admits a viscosity solution v and the optimal consumption policy can be represented in a feedback from the optimality conditions in the HJB equation.

This paper is organized as follows. In Section 2, we transform the two-dimensional HJB equation (6) associated with the stochastic Ramsey model. In Section 3, we show the existence of viscosity solution of the transformed HJB equation. In Section 4, a synthesis of the optimal consumption policy is presented in the feedback from the optimality conditions. Finally, Section 5 concludes with some remarks.

2. Transformed Hamilton-Jacobi-Bellman Equation

In order to transform the HJB equation (6) to one-dimensional second-order differential equation, that is, from the two-dimensional state space form (one state K for capital stock and the other state L for labor force), it has been transformed to a one-dimensional form, for ($x=K/L$) the ratio of capital to labor. Let us consider the solution $u(K,L)$ of (6) of the form

$$u(K,L)=v\left(\frac{K}{L}\right), \quad L>0. \tag{7}$$

Clearly

$$\begin{aligned} &Lu_K=v_K, \qquad Lu_L=-Kv_K,\\ &L^2u_{LL}=K^2v_{KK}+2Kv_K. \end{aligned} \tag{8}$$

Setting $x = K/L$ and substituting these above in (6), we have the HJB equation of v of the following form:

$$-\rho v(x) + \frac{1}{2}\sigma^2 x^2 v''(x) + \left(f(x) - \tilde{\lambda} x\right) v'(x) + U^*\left(x, v'(x)\right) = 0, \quad v(0) = 0, \quad x > 0, \tag{9}$$

where $\tilde{\lambda} = n + \lambda - \sigma^2$, $f(x) = F(x, 1)$, and $U^*(x, q) = \max_{0 \le c \le 1}\{U(cx) - cxq\}, q \in \mathbf{R}$.

We found that (9) is the transformed HJB equation associated with the stochastic utility consumption problem so as to maximize

$$\tilde{J}(c) = E\left[\int_0^T e^{-\rho t} U(c_t x_t)\, dt\right] \tag{10}$$

over the class $c \in \tilde{\mathscr{A}}$, subject to

$$dx_t = \left[f(x_t) - \tilde{\lambda} x_t - c_t\right] dt - \sigma x_t dW_t, \quad x_0 = x, \ x \ge 0, \tag{11}$$

where $c \in \tilde{\mathscr{A}}$ denotes the class $\mathscr{A}$ with $\{x_t\}$ replacing $\{K_t\}$. We choose $\delta > 0$ and rewrite (9) as

$$-\left(\rho + \frac{1}{\delta}\right) v(x) + \frac{1}{2}\sigma^2 x^2 v''(x) + \left(f(x) - \tilde{\lambda} x\right) v'(x) + U^*\left(x, v'(x)\right) + \frac{1}{\delta} v(x) = 0, \quad v(0) = 0, \quad x > 0. \tag{12}$$

The value function can be defined as a function whose value is the maximum value of the objective function of the consumption problem, that is,

$$V(x_t) = \sup_{c \in \tilde{\mathscr{A}}} E\left[\int_0^\tau e^{-(\rho + 1/\delta)t} \left\{U(c_t x_t) + \frac{1}{\delta} v(x_t)\right\} dt\right], \tag{13}$$

where $\tau = \tau(x) = \inf\{t \ge 0 : x_t = 0\}$ and $c = \{c_t\}$ is the element of the class $\tilde{\mathscr{A}}$ consisting of $\mathscr{F}_t$ progressively measurable processes such that

$$\int_0^t c_s ds < \infty \quad \text{a.s. } \forall t \ge 0. \tag{14}$$

3. Viscosity Solutions

In this section, we will show the existence results on the viscosity solution v of the HJB equation (9).

3.1. Definition. Let $v \in C[0, \infty)$ and $v(0) = 0$. Then v is called a viscosity solution of the reduced (one-dimensional) HJB equation (9) if the following relations hold:

$$\begin{aligned} &-\rho v(x) + \frac{1}{2}\sigma^2 x^2 q + p\left(f(x) - \tilde{\lambda} x\right) + U^*(x, p) \ge 0, \quad \forall (p, q) \in J^{2,+} v(x), \ \forall x > 0, \\ &-\rho v(x) + \frac{1}{2}\sigma^2 x^2 q + p\left(f(x) - \tilde{\lambda} x\right) + U^*(x, p) \le 0, \quad \forall (p, q) \in J^{2,-} v(x), \ \forall x > 0, \end{aligned} \tag{15}$$

where $J^{2,+}$ and $J^{2,-}$ are defined by

$$\begin{aligned} J^{2,+} v(x) &= \left\{ (p, q) \in \mathbf{R}^2 : \limsup_{\tilde{x} \to x} \frac{v(\tilde{x}) - v(x) - p(\tilde{x} - x) - (1/2) q |\tilde{x} - x|^2}{|\tilde{x} - x|^2} \le 0 \right\}, \\ J^{2,-} v(x) &= \left\{ (p, q) \in \mathbf{R}^2 : \liminf_{\tilde{x} \to x} \frac{v(\tilde{x}) - v(x) - p(\tilde{x} - x) - (1/2) q |\tilde{x} - x|^2}{|\tilde{x} - x|^2} \ge 0 \right\}. \end{aligned} \tag{16}$$

We assume that

$$\rho + \tilde{\lambda} > 0, \tag{17}$$

$$-\rho + \left(f'(0) + \left|\tilde{\lambda}\right|\right) + \frac{1}{2}\sigma^2 < 0. \tag{18}$$

Take $0 < \alpha < \rho$, and we choose $P_1 > 0$ by concavity such that

$$f(x) - \tilde{\lambda} x \le P_1. \tag{19}$$

Taking sufficiently large $P_0 > P_1$, we observe by (2) and (19) that $\phi(t, x) = e^{\tau - t}(x + P_0)$ fulfills

$$\begin{aligned} &-\alpha\phi(x) + \frac{1}{2}\sigma^2 x^2 \phi''(x) + \left(f(x) - \tilde{\lambda} x\right)\phi'(x) + U^*\left(x, \phi'(x)\right) \\ &\quad \le e^{\tau - t}\left(-x - P_0 + P_1\right) + Uo\left(U'\right)^{-1} e^{\tau - t} \\ &\quad < -x - P_0 + P_1 + Uo\left(U'\right)^{-1}(1) < 0, \end{aligned} \tag{20}$$

for some constant $P_0 > 0$.

Lemma 1. *One assumes* (2), (17), (11), *and* (20), *then the value function* $V(x)$ *fulfills*

$$0 \le V(x_t) \le \phi(x_t), \tag{21}$$

$$\sup_{c \in \tilde{\mathscr{A}}} E\left[e^{-(\rho + 1/\delta)\tau} |x(\tau) - \tilde{x}(\tau)|\right] \le |x - \tilde{x}| \tag{22}$$

for any stopping time τ, where $\{\tilde{x}(\tau)\}$ is the solution of (11) *to $c \in \mathscr{A}$ with $\tilde{x}(0) = \tilde{x}$.*

Proof. Itô's formula gives

$$\begin{aligned} 0 &\le e^{-(\rho+1/\delta)(t\wedge\tau)}\phi\left(x_{(t\wedge\tau)}\right) \\ &= \phi(x) + \int_0^{t\wedge\tau} e^{-(\rho+1/\delta)s} \\ &\quad \times \left\{ -\left(\rho+\frac{1}{\delta}\right)\phi(x_s) \right. \\ &\quad + \left(f(x) - \tilde{\lambda}x_s - c_s x_s\right)\phi'(x_s) \\ &\quad \left. + \frac{1}{2}\sigma^2 x_s^2 \phi''(x_s)\right\} ds \\ &\quad - \int_0^{t\wedge\tau} e^{-(\rho+1/\delta)s}\sigma x_s \phi'(x_s)\, dW_s. \end{aligned} \tag{23}$$

Since

$$\begin{aligned} E\left[\int_0^t \left|x_s\phi'(x_s)\right|^2 ds\right] &\le e^{2(\tau-t)} E\left[\int_0^t \left|x_s\right|^2 ds\right] \\ &\le e^{2(\tau-t)} E\left[\int_0^t \left(1+\left|x_s\right|^2\right)\, ds\right] \\ &= e^{2(\tau-t)}t + e^{2(\tau-t)} E\left[\int_0^t \left|x_s\right|^2 ds\right], \end{aligned} \tag{24}$$

and by considering $c_t = 0$, for all $t \ge 0$, then (11) becomes

$$d\check{x}_t = \left[f(\check{x}_t) - \tilde{\lambda}\check{x}_t\right]dt - \sigma\check{x}_t dw_t, \quad \check{x}_0 = x,\ x > 0, \tag{25}$$

by the comparison theorem of Ikeda and Watanabe [29]; we see that $0 < x_t \le \check{x}_t$, for all $t \ge 0$. Hence, by applying the existence and uniqueness theorem for (11), we have

$$E\left[\int_0^t \left|x_s\right|^2 ds\right] < \infty. \tag{26}$$

Therefore, from (24), we have

$$E\left[\int_0^t \left|x_s\phi'(x_s)\right|^2 ds\right] < \infty, \tag{27}$$

which yields that $\int_0^t e^{-(\rho+1/\delta)s}\sigma x_s\phi'(x_s)dW_s$ is a martingale, and again by (11), we can take sufficiently small $\delta > 0$ such that

$$\begin{aligned} &E\left[\sup_t \int_0^t e^{-(\rho+1/\delta)s}\sigma x_s\phi'(x_s)\, dW_s\right] \\ &\quad \le 2|\sigma| E\left[\sup_t \int_0^\infty e^{-(\rho+1/\delta)s} x_s^2 ds\right]^{1/2} < \infty. \end{aligned} \tag{28}$$

Hence,

$$E\left[\int_0^{t\wedge\tau} e^{-(\rho+1/\delta)s}\sigma x_s\phi'(x_s)\, dw_s\right] = 0. \tag{29}$$

Therefore, by (20), (11) and taking expectation on both sides of (23), we obtain

$$E\left[\int_0^{t\wedge\tau} e^{-(\rho+1/\delta)s}\left\{U(c_s x_s) + \frac{1}{\delta}\nu(x_s)\right\} ds\right] \le \phi(x), \tag{30}$$

from which we deduce (21).

We set $z_t = x_t - \tilde{x}_t$ and by (11), it is clear that

$$\begin{aligned} dz_t &= d\left(x_t - \tilde{x}_t\right) = \left[f(x_t) - f(\tilde{x}_t) - \tilde{\lambda}\left(x_t - \tilde{x}_t\right)\right]dt \\ &\quad - \sigma\left(x_t - \tilde{x}_t\right)dW_t. \end{aligned} \tag{31}$$

Since by (3), $f(z)$ is Lipschitz continuous and concave and $f(0) = 0$, then we have

$$dz_t \le \left(f'(0) + \left|\tilde{\lambda}\right|\right)z_t dt - \sigma z(t)\, dW_t, \quad z(0) = x - \tilde{x}. \tag{32}$$

Take $\mu > 0$ such that $g_\mu(z) = (z^2+\mu)^{1/2}$, and we can find from (18) and (20) that

$$\begin{aligned} &-\left(\rho+\frac{1}{\delta}\right)g_\mu(z) + \frac{1}{2}\sigma^2 z^2 g_\mu''(z) + \left(f'(0) + \left|\tilde{\lambda}\right|\right)\left|z g_\mu'(z)\right| \\ &\quad \le \left(z^2+\mu\right)^{1/2}\left\{-\left(\rho+\frac{1}{\delta}\right) + \frac{1}{2}\sigma^2 \right. \\ &\qquad \left. + \left(f'(0) + \left|\tilde{\lambda}\right|\right)\right\} < 0, \quad \forall z \in \mathbf{R}. \end{aligned} \tag{33}$$

Using Itô's formula and by (20), we have

$$\begin{aligned} &E\left[e^{-(\rho+1/\delta)\tau}g_\mu(z_\tau)\right] \\ &= g_\mu(x-\tilde{x}) \\ &\quad + E\left[\int_0^\tau e^{-(\rho+1/\delta)s} \right. \\ &\qquad \times \left\{-\left(\rho+\frac{1}{\delta}\right)g_\mu(z_s)\right. \\ &\qquad + \left(f(x_s) - f(\tilde{x}_s) - \tilde{\lambda}z_s\right)g_\mu'(z_s) \\ &\qquad \left. + \frac{1}{2}\sigma^2 z_s^2 g_\mu''(z_s)\right\} ds \\ &\qquad \left. - \int_0^\tau e^{-(\rho+1/\delta)s}\sigma z_s g_\mu'(z_s)\, dW_s\right] \\ &\le g_\mu(x-\tilde{x}). \end{aligned} \tag{34}$$

Letting $\delta \to 0$, and by Fatou's lemma, we obtain

$$E\left[e^{-(\rho+1/\delta)\tau}\left|z_\tau\right|\right] \le \left|x-\tilde{x}\right|, \tag{35}$$

which implies (22). □

Theorem 2. *One assumes* (2), (3), (17), *and* (18), *then the value function is a viscosity solution of the reduced (one-dimension) HJB equation* (9) *such that $0 \le \nu(x) \le \phi(x)$.*

Proof. Following (13) and (21) we have

$$v(x) = \sup_{c\in\widetilde{\mathscr{A}}} E\left[\int_0^{\tau} e^{-(\rho+1/\delta)t}\left\{U(c_t x_t) + \frac{1}{\delta}v(x_t)\right\}dt\right] \le \phi(x), \quad \forall x \ge 0, \tag{36}$$

for any stopping time τ. By (13) and for any $\varepsilon > 0$, there exists $c \in \widetilde{\mathscr{A}}$ such that

$$\begin{aligned} v(x) - \varepsilon &< E\left[\int_0^{\tau} e^{-(\rho+1/\delta)t}\left\{U(c_t x_t) + \frac{1}{\delta}v(x_t)\right\}dt\right] \\ &= E\left[\int_0^{\tau} e^{-(\rho+1/\delta)t}U(c_t x_t)\,dt\right] \\ &\quad + E\left[\int_0^{\tau} e^{-(\rho+1/\delta)t}\frac{1}{\delta}v(x_t)\,dt\right]. \end{aligned} \tag{37}$$

Since $f(z)$ is Lipschitz continuous, it follows that

$$\begin{aligned} dz_t &= \left[f(x_t) - f(\widetilde{x}_t) - \widetilde{\lambda}(x_t - \widetilde{x}_t)\right]dt - \sigma(x_t - \widetilde{x}_t)\,dW_t \\ &\le \left(f'(0) + \left|\widetilde{\lambda}\right|\right)z_t dt - \sigma z_t dW_t, \quad z(0) = x - \widetilde{x}. \end{aligned} \tag{38}$$

By (11), we can consider that $\widetilde{z}_t$ is the solution of

$$d\widetilde{z}_t = \left(f'(0) + \left|\widetilde{\lambda}\right|\right)\widetilde{z}_t dt - \sigma\widetilde{z}_t dW_t, \quad \widetilde{z}(0) = x > 0. \tag{39}$$

So by the comparison theorem Ikeda and Watanabe [29], we have

$$\tau_z \downarrow \widetilde{\tau}, \quad z_t \le \widetilde{z}_t \downarrow 0, \quad \text{a.s. as } z \downarrow 0. \tag{40}$$

Since $E[\sup_{0\le t\le L}\widetilde{z}_t^2] < \infty$ for all $L > 0$, now by (11) we have

$$E\left[z_{\tau\wedge L}\right] = z + E\left[\int_0^{\tau\wedge L}\left\{f(z_t) - \widetilde{\lambda}z_t - c_t\right\}dt\right]. \tag{41}$$

Letting $z \downarrow 0$ and then $L \to 0$, we obtain

$$E\left[\int_0^{\widetilde{\tau}} c_t dt\right] = E\left[\int_0^{\widetilde{\tau}}\left\{f(0) - c_t\right\}dt\right] \ge 0, \tag{42}$$

so that

$$E\left[\int_0^{\widetilde{\tau}} e^{-(\rho+1/\delta)t}U(c_t x_t)\,dt\right] = 0. \tag{43}$$

Passing to the limit to (37) and applying (43), we obtain

$$\begin{aligned} v(0+) - \varepsilon &< E\left[\int_0^{\infty} e^{-(\rho+1/\delta)t}\frac{1}{\delta}v(0+)\,dt\right] \\ &= \frac{v(0+)}{\rho\delta + 1}, \end{aligned} \tag{44}$$

which implies $v(0+) = 0$. Thus, $v \in C[0,\infty)$. So by the standard stability results of Fleming and Soner [27], we deduce that v is a viscosity solution of (9). □

4. Optimal Consumption Policy

Under the assumption (1) and (2), Lemma 3 has revealed that the value function of the representative household assets must approach zero as time approaches infinity.

Lemma 3. *One assumes* (2), (3), *and* (17). *Then for any* $(c_t) \in \mathscr{A}$. *One has*

$$\liminf_{t\to\infty} E\left[e^{-\rho t}u(K_t, L_t)\right] = 0. \tag{45}$$

Proof. By (17) and (18), we take $\mu \in (0,\rho)$ such that

$$-\mu + \frac{1}{2}\sigma^2 + \left(f'(0) + \left|\widetilde{\lambda}\right|\right) - \widetilde{\lambda} < 0. \tag{46}$$

Take $\mu > 0$ and $x = K/L > 0$ such that $g_\mu(x) = (x^2 + \mu)^{1/2}$, and by (33) and (46)

$$\begin{aligned} &-\mu g_\mu(x) + \frac{1}{2}\sigma^2 x^2 g_\mu''(x) + \left(f'(0) + \left|\widetilde{\lambda}\right|\right)\left|x g_\mu'(x)\right| \\ &\quad \le \left(x^2 + \mu\right)^{1/2}\left\{-\mu + \frac{1}{2}\sigma^2 + \left(f'(0) + \left|\widetilde{\lambda}\right|\right)\right\} < 0. \end{aligned} \tag{47}$$

Setting $\Phi(K,L) = A(K/L)^\gamma$, where $A, \gamma > 0$, we have by (6) and (47)

$$\begin{aligned} &-\mu\Phi(K,L) + \frac{1}{2}\sigma^2 L^2\Phi_{LL} + nL\Phi_L + \Phi_K(F(K,L) - \lambda K) \\ &\quad + U^*(L\Phi_K) < 0 \quad K, L > 0. \end{aligned} \tag{48}$$

By Itô's formula and (48), we obtain

$$\begin{aligned} &e^{-\rho t}\Phi(K_t, L_t) \\ &\quad = \Phi(K,L) + \int_0^t e^{-\rho s} \\ &\qquad \times\left\{(-\rho + \mu)\Phi\right. \\ &\qquad\quad + \Phi_K(F(K,L) - \lambda K - c_s L) + nL\Phi_L \\ &\qquad\quad \left. + \frac{1}{2}\sigma^2 L^2\Phi_{LL} - \mu\Phi\right\}\Big|_{(K=K_s, L=L_s)} ds \end{aligned}$$

$$\begin{aligned}
&-\int_0^t e^{-\rho s}\sigma L_s \Phi_L dw_s \\
&\le \Phi(K,L) + \int_0^t e^{-\rho s}\{(-\rho+\mu)\Phi - c_s L\Phi_K \\
&\qquad -U^*(L\Phi_K)\}|_{(K=K_s, L=L_s)} ds \\
&-\int_0^t e^{-\rho s}\sigma L_s \Phi_L dw_s \\
&\le \Phi(K,L) + \int_0^t e^{-\rho s}\{(-\rho+\mu)\Phi \\
&\qquad -U(c_s)\}|_{(K=K_s, L=L_s)} ds \\
&-\int_0^t e^{-\rho s}\sigma L_s \Phi_L dw_s, \\
&0 \le E\left[e^{-\rho t}\Phi(K_t, L_t)\right] \le \Phi(K,L) \\
&+ E\left[\int_0^t e^{-\rho s}(-\rho+\mu)\Phi(K_s, L_s) ds\right].
\end{aligned} \tag{49}$$

Letting $t \to \infty$, we have

$$E\left[\int_0^\infty e^{-\rho t}\Phi(K_t, L_t) dt\right] \le \frac{1}{\rho-\mu}\Phi(K,L) < \infty, \tag{50}$$

which implies $\liminf_{t\to\infty} E[e^{-\rho t}\Phi(K_t, L_t)] = 0$. By (21), we have

$$u(K,L) \le \Phi(K,L), \tag{51}$$

which completes the proof. □

We give a synthesis of the optimal policy $c^* = \{c_t^*\}$ for the optimization problem (4) subject to (5).

Theorem 4. *Under* (2) *and* (3), *there exists a unique solution* $K_t^* \ge 0$ *of*

$$dK_t^* = [F(K_t^*, L_t) - \lambda K_t^* - c_t^* K_t^*]\, dt \quad K_0^* = K,\ K > 0, \tag{52}$$

and the optimal consumption policy $c_t^* \in \mathscr{A}$ *is given by*

$$c_t^* = g\left(\frac{K_t}{L_t}, L_t u_K(K_t^*, L_t)\right). \tag{53}$$

Proof. Let us consider $G(K,L) = F(K_t, L_t) - \lambda K_t - g(K_t/L_t, L_t u_K(K_t, L_t))K_t$ and since $F(K,L)$. $g(K_t/L_t, L_t u_K(K_t^*, L_t))$ are continuous and $G(0,L) = 0$, there exists an $\mathscr{F}_t$ progressively measurable solution κ_t of

$$d\kappa_t = G(K_t, L_t)\, dt, \quad \kappa_0 = K > 0. \tag{54}$$

Now we shall show $K_t^* \ge 0$ for all $t \ge 0$ a.s. Suppose $0 < v'(0+) < \infty$. By (9), we have $v'(x) \ge 0$ for all $x > 0$, since $U^*(v'(x)) = \infty$ if $v'(x) < 0$. Moreover, by L'Hospital's rule this gives

$$\begin{aligned}
\lim_{x\to 0+} x^2 v''(x) &= \lim_{x\to 0+} x^2 v''(x) + 2xv'(x) \\
&= \lim_{x\to 0+} \frac{x^2 v'(x)}{x} = 0.
\end{aligned} \tag{55}$$

Letting $x \to 0+$ in (9), we have $U^*(v'(0+)) = 0$, and this is contrary with (2). Therefore, we get $v'(0+) = \infty$, which implies $g(0, Lu_K(0+, L)) = 0$. We note by the concavity of F that $G(K,L) \le C_1 K + C_2 L$ for C_1, $C_2 > 0$. Then applying the comparison theorem to (54) and

$$d\widetilde{K}_t = \left(C_1\widetilde{K}_t + C_2 L_t\right) dt, \quad \widetilde{K}_0 = K > 0, \tag{56}$$

we obtain $0 \le K_t^* \le \widetilde{K}_t$ for all $t \ge 0$. Further, in case $\tau_K^* = \inf\{t \ge 0 : K_t^* = 0\} = \tau_\kappa < \infty$, we have at $t = \tau_K^*$

$$\frac{dK_t^*}{dt} = F(K_t^*, L_t) - \lambda K_t^* - g\left(\frac{K_t}{L_t}, L_t u_K(K_t^*, L_t)\right) L_t = 0. \tag{57}$$

Therefore, K_t^* solves (52) and $K_t^* \ge 0$. To prove uniqueness, let $K_i^*(t)$, $i = 1, 2$, be two solutions of (52). Then $K_1^*(t) - K_2^*(t)$ satisfies

$$\begin{aligned}
d(K_1^*(t) - K_2^*(t)) = \Bigg[&(F(K_1^*(t), L_t) - F(K_2^*(t), L_t)) \\
&-\lambda(K_1^*(t) - K_2^*(t)) \\
&-\left(g\left(\frac{K_t}{L_t}, L_t u_K(K_t^*, L_t)\right) L_t\right. \\
&\left.-g\left(\frac{K_t}{L_t}, L_t u_K(K_t^*, L_t)\right) L_t\right)\Bigg] dt,
\end{aligned} \tag{58}$$

$K_1^*(0) - K_2^*(0) = 0$. We have

$$\begin{aligned}
&d(K_1^*(t) - K_2^*(t))^2 \\
&\quad = 2(K_1^*(t) - K_2^*(t))\, d(K_1^*(t) - K_2^*(t)) \\
&\quad = 2(K_1^*(t) - K_2^*(t)) \\
&\qquad \times \Bigg[(F(K_1^*(t), L_t) - F(K_2^*(t), L_t)) \\
&\qquad -\lambda(K_1^*(t) - K_2^*(t)) \\
&\qquad -\left(g\left(\frac{K_t}{L_t}, L_t u_K(K_t^*, L_t)\right) L_t\right. \\
&\qquad \left.-g\left(\frac{K_t}{L_t}, L_t u_K(K_t^*, L_t)\right) L_t\right)\Bigg] dt.
\end{aligned} \tag{59}$$

Note that the function $x \rightarrow -g(K_t/L_t, L_t u_K(K_t^*, L_t))L_t$ is decreasing. Hence,

$$\begin{aligned}
&\left(K_1^*(t) - K_2^*(t)\right)^2 \\
&\quad \leq 2\int_0^t \left(K_1^*(s) - K_2^*(s)\right) \\
&\qquad \times \left[\left(F\left(K_1^*(s), L_s\right) - F\left(K_2^*(s), L_s\right)\right)\right. \\
&\qquad\quad \left. -\lambda\left(K_1^*(s) - K_2^*(s)\right)\right] ds \\
&\quad \leq 2\left(F'(0) + \left|\tilde{\lambda}\right|\right)\int_0^t \left(K_1^*(s) - K_2^*(s)\right)^2 ds.
\end{aligned} \tag{60}$$

By Gronwall's lemma, we have

$$K_1^*(t) = K_2^*(t), \quad \forall t > 0. \tag{61}$$

So, the uniqueness of (52) holds.

Now by (6), (52), and Itô's formula, we have

$$\begin{aligned}
&e^{-\rho t} u\left(K_t^*, L_t\right) \\
&\quad = u(K, L) + \int_0^t e^{-\rho s} \\
&\qquad \times \Big\{ -\rho u(K, L) + u_K(K, L) \\
&\qquad\quad \times \left(F(K, L) - \lambda K - c_s^* K\right) \\
&\qquad\quad + nL u_L(K, L) \\
&\qquad\quad + \frac{1}{2}\sigma^2 L^2 u_{LL}(K, L)\Big\}\Big|_{(K=K_s^*, L=L_s)} ds \\
&\qquad + \int_0^t e^{-\rho s}\sigma L_s u_L(K, L)\, dW_s.
\end{aligned} \tag{62}$$

By the HJB equation (6), we have

$$\begin{aligned}
e^{-\rho t} u\left(K_t^*, L_t\right) = u(K, L) &- \int_0^t e^{-\rho s} U\left(\frac{c_s^* K_s^*}{L_s}\right) ds \\
&+ \int_0^t e^{-\rho s}\sigma L_s u_L(K, L)\, dW_s,
\end{aligned} \tag{63}$$

from which

$$\begin{aligned}
&E\left[e^{-\rho(t\wedge\tau_n)} u\left(K_{t\wedge\tau_n}^*, L_{t\wedge\tau_n}\right)\right] \\
&\quad + E\left[\int_0^{t\wedge\tau_n} e^{-\rho s} U\left(\frac{c_s^* K_s^*}{L_s}\right) ds\right] = u(K, L),
\end{aligned} \tag{64}$$

where $\{\tau_n\}$ is a sequence of localizing stopping times for the local martingale. From (11), (51), and Doob's inequalities for martingales, it follows that

$$\begin{aligned}
&E\left[\sup_n e^{-\rho(t\wedge\tau_n)} u\left(K_{t\wedge\tau_n}^*, L_{t\wedge\tau_n}\right)\right] \\
&\quad \leq E\left[\sup_{0\leq r\leq t} e^{-\alpha r}\phi\left(K_{(r)}^*, L_{(r)}\right)\right] \\
&\quad \leq \phi(K, L) + E\left[\sup_{0\leq r\leq t}\left|\int_0^r e^{-\alpha t}\sigma K_t^*\, dW_t\right|\right] \\
&\quad \leq \phi(K, L) + 2|\sigma| E\left[\int_0^t \left(e^{-\alpha t} K_t^*\right)^2 dt\right]^{1/2} \\
&\quad \leq \phi(K, L) + 2|\sigma| E\left[\int_0^t \widetilde{K}_t^2\, dt\right]^{1/2}.
\end{aligned} \tag{65}$$

Letting $n \rightarrow \infty$ and $t \rightarrow \infty$, hence, we obtain by the dominated convergence theorem

$$\begin{aligned}
&E\left[e^{-\rho\tau^*} u\left(K_{\tau^*}^*, L_{\tau^*}\right)\right] \\
&\quad + E\left[\int_0^{\tau^*} e^{-\rho s} U\left(\frac{c_s^* K_s^*}{L_s}\right) ds\right] = u(K, L).
\end{aligned} \tag{66}$$

We deduce by Lemma 3

$$J(c^*) = E\left[\int_0^{\tau^*} e^{-\rho s} U\left(\frac{c_s^* K_s^*}{L_s}\right) ds\right] = u(K, L). \tag{67}$$

Following the same calculation as above, we have

$$\begin{aligned}
&e^{-\rho t} u\left(K_t, L_t\right) \\
&\quad = u(K, L) + \int_0^t e^{-\rho s} \\
&\qquad \times \Big\{ -\rho u(K, L) \\
&\qquad\quad + u_K(K, L)\left(F(K, L) - \lambda K - c_s K\right) \\
&\qquad\quad + nL u_L(K, L)\frac{1}{2}\sigma^2 \\
&\qquad\quad \times L^2 u_{LL}(K, L)\Big\}\Big|_{(K=K_s, L=L_s)} ds \\
&\qquad + \int_0^t e^{-\rho s}\sigma L_s u_L(K, L)\, dW_s.
\end{aligned} \tag{68}$$

Again by the HJB equation (6), we can obtain

$$0 \leq E\left[e^{-\rho\tau} u\left(K_\tau, L_\tau\right)\right] \leq u(K, L) - E\left[\int_0^\tau e^{-\rho s} U\left(\frac{c_s K_s}{L_s}\right) ds\right] \tag{69}$$

from which

$$J(c) = E\left[\int_0^\tau e^{-\rho s} U\left(\frac{c_s K_s}{L_s}\right) ds\right] \leq u(K, L) \tag{70}$$

for any $(c_t) \in \mathscr{A}$. The proof is complete. □

Remark 5. From the proof of Theorem 4, it follows that

$$\inf_{c\in\mathscr{A}} E\left[\int_s^T e^{-\rho t} U\left(\frac{c_t K_t}{L_t}\right) dt\right] = V(s, K, L). \quad (71)$$

Thus, under (2), we observe that the smooth solution of u of the HJB equation (6). Furthermore, let v be the solution of (9) on the entire domain $[0, T) \times (0, \infty)$ with $v(T, x) = 0$, $x > 0$. Setting $x = K/L$ and $u(t, K, L) = v(t, K/L)$, $K, L > 0$, by (8), we have that u satisfies (6). Therefore, we obtain the uniqueness of v.

5. Concluding Remarks

In this paper we have studied the optimal consumption problem of maximizing the expected discounted value of consumption utility in the context of one-sector neoclassical economic growth with Cobb-Douglas production function. We have derived a transformed (one-dimensional) Hamilton-Jacobi-Bellman equation associated with the optimization problem. By the technique of viscosity method we established the viscosity solution to the transformed (one-dimensional) Hamilton-Jacobi-Bellman equation. Finally we have derived the optimal consumption feedback form from the optimality conditions in the two-dimensional HJB equation.

Acknowledgment

The authors wish to thank the anonymous referees for their comments that have led to an improved version of this paper.

References

[1] R. C. Merton, "Lifetime portfolio selection under uncertainity: the continuous time case," *Review of Economics and Statistics*, vol. 51, no. 3, pp. 247–257, 1969.

[2] R. C. Merton, "An asymptotic theory of growth under uncertainty," *Review of Economic Studies*, vol. 42, no. 3, pp. 375–393, 1975.

[3] M. I. Kamien and N. L. Schwartz, *Dynamic Optimization, The Calculus of Variations and 0ptimal Control in Economics and Management*, North-Holland Publishing Co., Amsterdam, The Netherlands, 1981.

[4] H. K. Koo, "Consumption and portfolio selection with labor income: a continuous time approach," *Mathematical Finance*, vol. 8, no. 1, pp. 49–65, 1988.

[5] H. Morimoto and K. Kawaguchi, "Optimal exploitation of renewable resources by the viscosity solution method," *Stochastic Analysis and Applications*, vol. 20, no. 5, pp. 927–946, 2002.

[6] H. Morimoto, "Optimal consumption models in economic growth," *Journal of Mathematical Analysis and Applications*, vol. 337, no. 1, pp. 480–492, 2008.

[7] S. P. Zeldes, "Optimal consumption with stochastic income: deviations from certainty equivalence," *Quarterly Journal of Economics*, vol. 104, no. 2, pp. 275–298, 1989.

[8] Md. A. Baten and A. Sobhan, "Optimal consumption in a growth model with the CES production function," *Stochastic Analysis and Applications*, vol. 25, no. 5, pp. 1025–1042, 2007.

[9] H. Amilon and H.-P. Bermin, "Welfare effects of controlling labor supply: an application of the stochastic Ramsey model," *Journal of Economic Dynamics & Control*, vol. 28, no. 2, pp. 331–348, 2003.

[10] A. Bucci, C. Colapinto, M. Forster, and D. L. Torre, "On human capital and economic growth with random technology shocks," Departemental Working Paper 2008-36, University of Milan, Department of Economics, Milan, Italy, 2008.

[11] O. Posch, "Structural estimation of jump-diffusion processes in macroeconomics," *Journal of Econometrics*, vol. 153, no. 2, pp. 196–210, 2009.

[12] H. Roche, "Stochastic growth: a duality approach," *Journal of Economic Theory*, vol. 113, no. 1, pp. 131–143, 2003.

[13] F.-R. Chang, *Stochastic Optimization in Continuous Time*, Cambridge University Press, Cambridge, UK, 2004.

[14] A. G. Malliaris, Malliare, and Haltiwanger, *Stochastic Methods in Economics and Finance*, North-Holland Publishing Co., Amsterdam, The Netherlands, 1982.

[15] S. J. Turnovsky, "Government policy in a stochastic growth model with elastic labor supply," *Journal of Public Economic Theory*, vol. 2, no. 4, pp. 389–433, 2000.

[16] S. J. Turnovsky, *Methods of Macroeconomic Dynamics*, MIT Press, Cambridge, Mass, USA, 2nd edition, 2000.

[17] K. Walde, "Endogenous growth cycles," *International Economic Review*, vol. 46, no. 3, pp. 867–894, 2005.

[18] K. Walde, *Applied Intertemporal Optimization. Lecture Notes*, University of Mainz, Mainz, Germany, 2009.

[19] K. Walde, "Production technologies in stochastic continuous time models," CESifo Working Paper 2831, CESifo Group Munich, Munchen, Germany, 2009.

[20] W. Smith, "Inspecting the mechanism exactly: a closed-form solution to a stochastic growth model," vol. 7, no. 1, p. 30, 2007.

[21] F. Bourguignon, "A particular class of continuous-time stochastic growth models," *Journal of Economic Theory*, vol. 9, no. 2, pp. 141–158, 1974.

[22] B. S. Jensen and M. Richter, "Stochastic one-sector and two-sector growth models in continuous time," in *Stochastic Economic Dynamics, Part 5*, B. S. Jensen and T. Palokangas, Eds., pp. 167–216, Business School Press, Copenhagen, Denmark, 2007.

[23] B. Oksendal, *A Universal Optimal Consumption Rate for an Insider*, Pure Mathematics, Department of Mathematics, University of Oslo, Oslo, Norway, 2004.

[24] C. Blanchet-Scalliet, N. El Karoui, M. Jeanblanc, and L. Martellini, "Optimal investment decisions when time-horizon is uncertain," *Journal of Mathematical Economics*, vol. 44, no. 11, pp. 1100–1113, 2008.

[25] B. Bouchard and H. Pham, "Wealth-path dependent utility maximization in incomplete markets," *Finance and Stochastics*, vol. 8, no. 4, pp. 579–603, 2004.

[26] C. Blanchet-Scalliet, N. Karoui, M. Jeanblanc, and L. Martinelli, "Optimal investment and consumption decisions when time-horizon is uncertain," *Journal of Economics Dynamics and Control*, vol. 29, no. 10, pp. 1737–1764, 2005.

[27] W. H. Fleming and H. M. Soner, *Controlled Markov Processes and Viscosity Solutions*, Springer-Verlag, New York, NY, USA, 1993.

[28] H. M. Soner, "Controlled Markov processes, viscosity solutions and applications to mathematical finance," in *Viscosity Solutions and Applications*, vol. 1660 of *Lecture Notes in Mathematics*, pp. 134–185, Springer, Berlin, Germany, 1997.

[29] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, North-Holland, Amsterdam, The Netherlands, 1981.