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Research Article

Some Elementary Aspects of Means

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We raise several elementary questions pertaining to various aspects of means. These questions refer to both known and newly introduced families of means, and include questions of characterizations of certain families, relations among certain families, comparability among the members of certain families, and concordance of certain sequences of means. They also include questions about internality tests for certain mean-looking functions and about certain triangle centers viewed as means of the vertices. The questions are accessible to people with no background in means, and it is also expected that these people can seriously investigate, and contribute to the solutions of, these problems. The solutions are expected to require no more than simple tools from analysis, algebra, functional equations, and geometry.

1. Definitions and Terminology

In all that follows, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{J}$ denotes an interval in $\mathbb{R}$.

By a data set (or a list) in a set S, we mean a finite subset of S in which repetition is allowed. Although the order in which the elements of a data set are written is not significant, we sometimes find it convenient to represent a data set in S of size n by a point in S^n, the cartesian product of n copies of S.

We will call a data set $A = (a_1, \dots, a_n)$ in $\mathbb{R}$ *ordered* if $a_1 \le \cdots \le a_n$. Clearly, every data set in $\mathbb{R}$ may be assumed ordered.

A *mean* of k variables (or a *k-dimensional mean*) on $\mathbb{J}$ is defined to be any function $\mathscr{M} : \mathbb{J}^k \to \mathbb{J}$ that has the *internality* property

$$\min \{a_1, \dots, a_k\} \le \mathscr{M}(a_1, \dots, a_k) \le \max \{a_1, \dots, a_k\} \quad (1)$$

for all a_j in $\mathbb{J}$. It follows that a mean $\mathscr{M}$ must have the property $\mathscr{M}(a, \dots, a) = a$ for all a in $\mathbb{J}$.

Most means that we encounter in the literature, and all means considered below, are also symmetric in the sense that

$$\mathscr{M}(a_1, \dots, a_k) = \mathscr{M}(a_{\sigma(1)}, \dots, a_{\sigma(n)}) \quad (2)$$

for all permutations σ on $\{1, \dots, n\}$, and *1-homogeneous* in the sense that

$$\mathscr{M}(\lambda a_1, \dots, \lambda a_k) = \lambda \mathscr{M}(a_{\sigma(1)}, \dots, a_{\sigma(n)}) \quad (3)$$

for all permissible $\lambda \in \mathbb{R}$.

If $\mathscr{M}$ and $\mathscr{N}$ are two k-dimensional means on $\mathbb{J}$, then we say that $\mathscr{M} \le \mathscr{N}$ if $\mathscr{M}(a_1, \dots, a_k) \le \mathscr{N}(a_1, \dots, a_k)$ for all $a_j \in \mathbb{J}$. We say that $\mathscr{M} < \mathscr{N}$ if $\mathscr{M}(a_1, \dots, a_k) < \mathscr{N}(a_1, \dots, a_k)$ for all $a_j \in \mathbb{J}$ for which $a_1, \dots, a_k$ are not all equal. This exception is natural since $\mathscr{M}(a, \dots, a)$ and $\mathscr{N}(a, \dots, a)$ must be equal, with each being equal to a. We say that $\mathscr{M}$ and $\mathscr{N}$ are *comparable* if $\mathscr{M} \le \mathscr{N}$ or $\mathscr{N} \le \mathscr{M}$.

A *distance* (or *a distance function*) on a set S is defined to be any function $d : S \times S \to [0, \infty)$ that is *symmetric* and *positive definite*, that is,

$$d(a,b) = d(b,a), \quad \forall a, b \in S,$$
$$d(a,b) = 0 \Longleftrightarrow a = b. \quad (4)$$

Thus a *metric* is a distance that satisfies the triangle inequality

$$d(a,b) + d(b,c) \ge d(a,c), \quad \forall a, b, c \in S, \quad (5)$$

a condition that we find too restrictive for our purposes.

2. Examples of Means

The *arithmetic*, *geometric*, and *harmonic* means of two positive numbers were known to the ancient Greeks; see [1, pp. 84–90]. They are usually denoted by $\mathscr{A}$, $\mathscr{G}$, and $\mathscr{H}$, respectively, and are defined, for $a, b > 0$, by

$$\mathscr{A}(a,b) = \frac{a+b}{2},$$
$$\mathscr{G}(a,b) = \sqrt{ab},$$
$$\mathscr{H}(a,b) = \frac{2}{1/a + 1/b} = \frac{2ab}{a+b}. \tag{6}$$

The celebrated inequalities

$$\mathscr{H}(a,b) < \mathscr{G}(a,b) < \mathscr{A}(a,b) \quad \forall a, b > 0 \tag{7}$$

were also known to the Greeks and can be depicted in the well-known figure that is usually attributed to Pappus and that appears in [2, p. 364]. Several other less well known means were also known to the ancient Greeks; see [1, pp. 84–90].

The three means above, and their natural extensions to any number n of variables, are members of a large two-parameter family of means, known now as the *Gini* means and defined by

$$G_{r,s}(x_1, \ldots, x_n) = \left(\frac{N_r(x_1, \ldots, x_n)}{N_s(x_1, \ldots, x_n)} \right)^{1/(r-s)}, \tag{8}$$

where N_j are the Newton polynomials defined by

$$N_j(x_1, \ldots, x_n) = \sum_{k=1}^{n} x_k^j. \tag{9}$$

Means of the type $G_{r,r-1}$ are known as *Lehmer*'s means, and those of the type $G_{r,0}$ are known as *Hölder* or *power* means. Other means that have been studied extensively are the *elementary symmetric polynomial* and *elementary symmetric polynomial ratio* means defined by

$$\left(\frac{\sigma_r}{C_r^n} \right)^{1/r}, \qquad \frac{\sigma_r / C_r^n}{\sigma_{r-1} / C_{r-1^n}}, \tag{10}$$

where σ_r is the rth elementary symmetric polynomial in n variables, and where

$$C_r^n = \binom{n}{r}. \tag{11}$$

These are discussed in full detail in the encyclopedic work [3, Chapters III and V].

It is obvious that the power means $\mathscr{P}_r$ defined by

$$\mathscr{P}_r(a_1, \ldots, a_n) = G_{r,0}(a_1, \ldots, a_n) = \left(\frac{a_1^r + \cdots + a_n^r}{n} \right)^{1/r} \tag{12}$$

that correspond to the values $r = -1$ and $r = 1$ are nothing but the harmonic and arithmetic means $\mathscr{H}$ and $\mathscr{A}$, respectively. It is also natural to set

$$\mathscr{P}_0(a_1, \ldots, a_n) = \mathscr{G}(a_1, \ldots, a_n) = (a_1 \ldots a_n)^{1/n}, \tag{13}$$

since

$$\lim_{r \to 0} \left(\frac{a_1^r + \cdots + a_n^r}{n} \right)^{1/r} = (a_1 \ldots a_n)^{1/n} \tag{14}$$

for all $a_1, \ldots, a_n > 0$.

The inequalities (7) can be written as $\mathscr{P}_{-1} < \mathscr{P}_0 < \mathscr{P}_1$. These inequalities hold for any number of variables and they follow from the more general fact that $\mathscr{P}_r(a_1, \ldots, a_n)$, for fixed $a_1, \ldots, a_n > 0$, is strictly increasing with r. Power means are studied thoroughly in [3, Chapter III].

3. Mean-Producing Distances and Distance Means

It is natural to think of the mean of any list of points in any set to be the point that is *closest* to that list. It is also natural to think of a point as closest to a list of points if the sum of its distances from these points is minimal. This mode of thinking associates means to distances.

If d is a distance on S, and if $A = (a_1, \ldots, a_n)$ is a data set in S, then a *d-mean* of A is defined to be any element of S at which the function

$$f(x) = \sum_{i=1}^{n} d(x, a_i) \tag{15}$$

attains its minimum. It is conceivable that (15) attains its minimum at many points, or nowhere at all. However, we shall be mainly interested in distances d on $\mathbb{J}$ for which (15) attains its minimum at a unique point x_A that, furthermore, has the property

$$\min\{a : a \in A\} \le x_A \le \max\{a : a \in A\} \tag{16}$$

for every data set A. Such a distance is called a *mean-producing* or a *mean-defining* distance, and the point x_A is called the *d-mean* of A or the *mean of A arising from the distance d* and will be denoted by $\mu_d(A)$. A mean $\mathscr{M}$ is called a *distance mean* if it is of the form μ_d for some distance d.

Problem Set 1. (1-a) Characterize those distances on $\mathbb{J}$ that are mean-producing.

(1-b) Characterize those pairs of mean producing distances on $\mathbb{J}$ that produce the same mean.

(1-c) Characterize distance means.

4. Examples of Mean-Producing Distances

If d_0 is the discrete metric defined on $\mathbb{R}$ by

$$d_0(a,b) = \begin{cases} 1 & \text{if } a \neq b, \\ 0 & \text{if } a = b, \end{cases} \tag{17}$$

then the function $f(x)$ in (15) is nothing but the number of elements in the given data set A that are different from x, and therefore every element having maximum frequency in A minimizes (15) and is hence a d_0-mean of A. Thus the discrete metric gives rise to what is referred to in statistics as "the" *mode* of A. Due to the nonuniqueness of the mode, the discrete metric is not a mean-producing distance.

Similarly, the usual metric $d = d_1$ defined on $\mathbb{R}$ by

$$d_1(a,b) = |a-b| \tag{18}$$

is not a mean-producing distance. In fact, it is not very difficult to see that if $A = (a_1, \ldots, a_n)$ is an ordered data set of even size $n = 2m$, then any number in the closed interval $[a_m, a_{m+1}]$ minimizes

$$\sum_{j=1}^{n} \left|x - a_j\right| \tag{19}$$

and is therefore a d_1-mean of A. Similarly, one can show that if A is of an odd size $n = 2m-1$, then a_m is the unique d_1-mean of A. Thus the usual metric on $\mathbb{R}$ gives rise to what is referred to in statistics as "the" *median* of A.

On the other hand, the distance d_2 defined on $\mathbb{R}$ by

$$d_2(a,b) = (a-b)^2 \tag{20}$$

is a mean-producing distance, although it is not a metric. In fact, it follows from simple derivative considerations that the function

$$\sum_{j=1}^{n} \left(x - a_j\right)^2 \tag{21}$$

attains its minimum at the unique point

$$x = \frac{1}{n}\left(\sum_{j=1}^{n} a_j\right). \tag{22}$$

Thus d_2 is a mean-producing distance, and the corresponding mean is nothing but the arithmetic mean.

It is noteworthy that the three distances that come to mind most naturally give rise to the three most commonly used "means" in statistics. In this respect, it is also worth mentioning that a fourth mean of statistics, the so-called *midrange*, will be encountered below as a very natural *limiting* distance mean.

The distances d_1 and d_2 (and in a sense, d_0 also) are members of the family d_p of distances defined by

$$d_p(a,b) = |a-b|^p. \tag{23}$$

It is not difficult to see that if $p > 1$, then d_p is a mean-producing distance. In fact, if $A = (a_1, \ldots, a_n)$ is a given data set, and if

$$f(x) = \sum_{j=1}^{n} \left|x - a_j\right|^p, \tag{24}$$

then

$$f''(x) = p(p-1)\sum_{j=1}^{n} \left|x - a_j\right|^{p-2} \geq 0, \tag{25}$$

with equality if and only if $a_1 = \cdots = a_n = x$. Thus f is convex and cannot attain its minimum at more than one point. That it attains its minimum follows from the continuity of $f(x)$, the compactness of $[a_1, a_n]$, and the obvious fact that $f(x)$ is increasing on $[a_n, \infty)$ and is decreasing on $(-\infty, a_1]$. If we denote the mean that d_p defines by μ_p, then $\mu_p(A)$ is the unique zero of

$$\sum_{j=1}^{n} \operatorname{sign}\left(x - a_j\right) \left|x - a_j\right|^{p-1}, \tag{26}$$

where $\operatorname{sign}(t)$ is defined to be 1 if t is nonnegative and -1 otherwise.

Note that no matter what $p > 1$ is, the two-dimensional mean μ_p arising from d_p is the arithmetic mean. Thus when studying μ_p, we confine our attention to the case when the number k of variables is greater than two. For such k, it is impossible in general to compute $\mu_p(A)$ in closed form.

Problem 2. It would be interesting to investigate comparability among $\{\mu_p : p > 1\}$.

It is highly likely that no two means μ_p are comparable.

5. Deviation and Sparseness

If d is a mean-producing distance on S, and if μ_d is the associated mean, then it is natural to define the d-deviation $\mathcal{D}_d(A)$ of a data set $A = (a_1, \ldots, a_n)$ by an expression like

$$\mathcal{D}_d(A) = \mu_d\left\{d\left(\mu_d(A), a_i\right) : 1 \leq i \leq n\right\}. \tag{27}$$

Thus if d is defined by

$$d(x,y) = (x-y)^2, \tag{28}$$

then μ_d is nothing but the arithmetic mean or ordinary *average* μ defined by

$$\mu = \mu(a_1, \ldots, a_n) = \frac{a_1 + \cdots + a_n}{n}, \tag{29}$$

and $\mathcal{D}_d$ is the (squared) standard deviation $\sigma^{(2)}$ given by

$$\sigma^{(2)}(a_1, \ldots, a_n) = \frac{|a_1 - \mu|^2 + \cdots + |a_n - \mu|^2}{n}. \tag{30}$$

In a sense, this provides an answer to those who are puzzled and mystified by the choice of the exponent 2 (and not any other exponent) in the standard definition of the standard deviation given in the right-hand side of (30). In fact, distance means were devised by the author in an attempt to remove that mystery. Somehow, we are saying that the ordinary average μ and the standard deviation $\sigma^{(2)}$ must be taken or discarded together, being both associated with the same distance d given in (28). Since few people question the sensibility of the definition of μ given in (29), accepting the standard definition of the standard deviation given in (30) as is becomes a *must*.

It is worth mentioning that choosing an exponent other than 2 in (30) would result in an *essentially* different notion of deviations. More precisely, if one defines $\sigma^{(k)}$ by

$$\sigma^{(k)}(a_1, \ldots, a_n) = \frac{|a_1 - \mu|^k + \cdots + |a_n - \mu|^k}{n}, \tag{31}$$

then $\sigma^{(k)}$ and $\sigma^{(2)}$ would of course be unequal, but more importantly, they would not be *monotone* with respect to each other, in the sense that there would exist data sets A and B with $\sigma^{(2)}(A) > \sigma^{(k)}(B)$ and $\sigma^{(2)}(A) < \sigma^{(k)}(B)$. Thus the choice of the exponent k in defining deviations is not as arbitrary as some may feel. On the other hand, it is (27) and not (31) that is the natural generalization of (30). This raises the following, expectedly hard, problem.

Problem 3. Let d be the distance defined by $d(x, y) = |x - y|^k$, and let the associated deviation $\mathscr{D}_d$ defined in (27) be denoted by $\mathscr{D}_k$. Is $\mathscr{D}_k$ monotone with respect to $\mathscr{D}_2$ for any $k \neq 2$, in the sense that

$$\mathscr{D}_k(A) > \mathscr{D}_k(B) \Longrightarrow \mathscr{D}_2(A) > \mathscr{D}_2(B)? \tag{32}$$

We end this section by introducing the notion of *sparseness* and by observing its relation with deviation. If d is a mean-producing distance on $\mathbb{J}$, and if μ_d is the associated mean, then the *d-sparseness* $\mathscr{S}_d(A)$ of a data set $A = (a_1, \ldots, a_n)$ in $\mathbb{J}$ can be defined by

$$\mathscr{S}_d(A) = \mu_d\left\{d\left(a_i, a_j\right) : 1 \le i < j \le n\right\}. \tag{33}$$

It is interesting that when d is defined by (28), the standard deviation coincides, up to a constant multiple, with the sparsenss. One wonders whether this pleasant property characterizes this distance d.

Problem Set 4. (4-a) Characterize those mean-producing distances whose associated mean is the arithmetic mean.

(4-b) If d is as defined in (28), and if d' is another mean-producing distance whose associated mean is the arithmetic mean, does it follow that $\mathscr{D}_{d'}$ and $\mathscr{D}_d$ are monotone with respect to each other?

(4-c) Characterize those mean-producing distances δ for which the deviation $\mathscr{D}_\delta(A)$ is determined by the sparseness $\mathscr{S}_\delta(A)$ for every data set A, and vice versa.

6. Best Approximation Means

It is quite transparent that the discussion in the previous section regarding the distance mean μ_p, $p > 1$, can be written in terms of best approximation in ℓ_p^n, the vector space $\mathbb{R}^n$ endowed with the p-norm $\|\cdots\|_p$ defined by

$$\left\|(a_1, \ldots, a_n)\right\|_p = \left(\sum_{j=1}^{n} \left|a_j\right|^p\right)^{1/p}. \tag{34}$$

If we denote by $\Delta = \Delta_n$ the line in $\mathbb{R}^n$ consisting of the points $(x_1, \ldots, x_n)$ with $x_1 = \cdots = x_n$, then to say that $a = \mu_p(a_1, \ldots, a_n)$ is just another way of saying that the point $(a, \ldots, a)$ is a best approximant in Δ_n of the point $(a_1, \ldots, a_n)$ with respect to the p-norm given in (34). Here, a point s_t in a subset S of a metric (or distance) space (T, D) is said to be a *best approximant* in S of $t \in T$ if $D(t, s_t) = \min\{D(t, s) : s \in S\}$. Also, a subset S of (T, D) is said to be *Chebyshev* if every t in T has exactly one best approximant in S; see [4, p. 21].

The discussion above motivates the following definition.

Definition 1. Let $\mathbb{J}$ be an interval in $\mathbb{R}$ and let D be a distance on $\mathbb{J}^n$. If the diagonal $\Delta(\mathbb{J}^n)$ of $\mathbb{J}^n$ defined by

$$\Delta(\mathbb{J}^n) = \{(a_1, \ldots, a_n) \in \mathbb{J}^n : a_1 = \cdots = a_n\} \tag{35}$$

is Chebyshev (with respect to D), then the n-dimensional mean M_D on $\mathbb{J}$ defined by declaring $M_D(a_1, \ldots, a_n) = a$ if and only if $(a, \ldots, a)$ is the best approximant of $(a_1, \ldots, a_n)$ in $\Delta(\mathbb{J}^n)$ is called the *Chebyshev* or *best approximation D-mean* or the *best approximation mean arising from D*.

In particular, if one denotes by M_p the best approximation n-dimensional mean on $\mathbb{R}$ arising from (the distance on $\mathbb{R}^n$ induced by) the norm $\|\cdots\|_p$, then the discussion above says that M_p exists for all $p > 1$ and that it is equal to μ_p defined in Section 4.

In view of this, one may also define M_∞ to be the best approximation mean arising from the ∞-norm of ℓ_∞^n, that is, the norm $\|\cdots\|_\infty$ defined on $\mathbb{R}^n$ by

$$\left\|(a_1, \ldots, a_n)\right\|_\infty = \max\left\{\left|a_j\right| : 1 \le j \le n\right\}. \tag{36}$$

It is not very difficult to see that $\mu_\infty(A)$ is nothing but what is referred to in statistics as the *mid-range* of A. Thus if $A = (a_1, \ldots, a_n)$ is an ordered data set, then

$$M_\infty(A) = \frac{a_1 + a_n}{2}. \tag{37}$$

In view of the fact that d_∞ cannot be defined by anything like (23) and μ_∞ is thus meaningless, natural question arises as to whether

$$M_\infty(A) = \lim_{p \to \infty} \mu_p(A) \quad \left(\text{or equivalently} = \lim_{p \to \infty} M_p(A)\right) \tag{38}$$

for every A. An affirmative answer is established in [5, Theorem 1]. In that theorem, it is also established that

$$\lim_{p \to q} \mu_p(A) \quad \left(\text{or equivalently } \lim_{p \to q} M_p(A)\right) = M_q(A) \tag{39}$$

for all q and all A. All of this can be expressed by saying that μ_p is continuous in p for $p \in (1, \infty]$ for all A.

We remark that there is no *obvious* reason why (38) should immediately follow from the well known fact that

$$\lim_{p \to \infty} \|A\|_p = \|A\|_\infty \tag{40}$$

for all points A in $\mathbb{R}^n$.

Problem Set 5. Suppose that δ_p is a sequence of distances on a set S that converges to a distance δ_∞ (in the sense that $\lim_{p\to\infty}\delta_p(a,b) = \delta_\infty(a,b)$ for all a, b in S). Let $T \subseteq S$.

(5-a) If T is Chebyshev with respect to each δ_p, is it necessarily true that T is Chebyshev with respect to δ_∞?

(5-b) If T is Chebyshev with respect to each δ_p and with respect to δ_∞ and if x_p is the best approximant in T of x with respect to δ_p and x_∞ is the best approximant in T of x with respect to δ_∞, does it follow that x_p converges to x_∞?

We end this section by remarking that if $M = M_d$ is the n-dimensional best approximation mean arising from a distance d on $\mathbb{J}^n$, then d is significant only up to its values of the type $d(u,v)$, where $u \in \Delta(\mathbb{J}^n)$ and $v \notin \Delta(\mathbb{J}^n)$. Other values of d are not significant. This, together with the fact that

$$\text{every mean is a best approximation mean arising from a metric,} \tag{41}$$

makes the study of best approximation means less interesting. Fact (41) was proved in an unduly complicated manner in [6], and in a trivial way based on a few-line set-theoretic argument in [7].

Problem 6. Given a mean $\mathscr{M}$ on $\mathbb{J}$, a metric D on $\mathbb{J}$ is constructed in [6] so that $\mathscr{M}$ is the best approximation mean arising from D. Since the construction is extremely complicated in comparison with the construction in [7], it is desirable to examine the construction of D in [6] and see what other nice properties (such as continuity with respect to the usual metric) D has. This would restore merit to the construction in [6] and to the proofs therein and provide raison d'être for the so-called *generalized* means introduced there.

7. Towards a Unique Median

As mentioned earlier, the distance d_1 on $\mathbb{R}$ defined by (23) does not give rise to a (distance) mean. Equivalently, the 1-norm $\|\cdots\|_1$ on $\mathbb{R}^n$ defined by (34) does not give rise to a (best approximation) mean. These give rise, instead, to the many-valued function known as *the median*. Thus, following the statistician's mode of thinking, one may set

$$\begin{aligned}\mu_1(A) = M_1(A) &= \text{ the median interval of } A \\ &= \text{ the set of all medians of } A.\end{aligned} \tag{42}$$

From a mathematician's point of view, however, this leaves a lot to be desired, to say the least. The feasibility and naturality of defining μ_∞ as the limit of μ_p as p approaches ∞ gives us a clue on how the median μ_1 may be defined. It is a pleasant fact, proved in [5, Theorem 4], that the limit of $\mu_p(A)$ (equivalently of $M_p(A)$) as p decreases to 1 exists for every $A \in \mathbb{R}^n$ and equals one of the medians described in (42). This limit can certainly be used as *the* definition of *the* median.

Problem Set 7. Let μ_p be as defined in Section 4, and let μ^* be the limit of μ_p as p decreases to 1.

(7-a) Explore how the value of $\mu^*(A)$ compares with the common practice of taking the median of A to be the midpoint of the median interval (defined in (42) for various values of A.

(7-b) Is μ^* continuous on $\mathbb{R}^n$? If not, what are its points of discontinuity?

(7-c) Given $A \in \mathbb{R}^n$, is the convergence of $\mu_p(A)$ (as p decreases to 1) to $\mu^*(A)$ monotone?

The convergence of $\mu_p(A)$ (as p decreases to 1) to $\mu^*(A)$ is described in [5, Theorem 4], where it is proved that the convergence is ultimately monotone. It is also proved in [5, Theorem 5] that when $n = 3$, then the convergence is monotone.

It is of course legitimate to question the usefulness of defining the median to be μ^*, but that can be left to statisticians and workers in relevant disciplines to decide. It is also legitimate to question the path that we have taken the limit along. In other words, it is conceivable that there exists, in addition to d_p, a sequence d'_p of distances on $\mathbb{R}$ that converges to d_1 such that the limit μ^{**}, as p decreases to 1, of their associated distance means μ'_p is not the same as the limit μ^* of μ_p. In this case, μ^{**} would have as valid a claim as μ^* to being *the* median. However, the naturality of d_p may help accepting μ^* as a most legitimate median.

Problem Set 8. Suppose that δ_p and δ'_p, $p \in \mathbb{N}$, are sequences of distances on a set S that converge to the distances δ_∞ and δ'_∞, respectively (in the sense that $\lim_{p\to\infty}\delta_p(a,b) = \delta_\infty(a,b)$ for all a, b in S, etc.).

(8-a) If each δ_p, $p \in \mathbb{N}$, is mean producing with corresponding mean m_p, does it follow that δ_∞ is mean producing? If so, and if the mean produced by δ_∞ is m_∞, is it necessarily true that m_p converges to m_∞?

(8-b) If δ_p and δ'_p, $p \in \mathbb{N} \cup \{\infty\}$, are mean producing distances with corresponding means m_p and m'_p, and if $m_p = m'_p$ for all $p \in \mathbb{N}$, does it follow that $m_\infty = m'_\infty$?

8. Examples of Distance Means

It is clear that the arithmetic mean is the distance mean arising from the the distance d_2 given by $d_2(a,b) = (a-b)^2$. Similarly, the geometric mean on the set of positive numbers is the distance mean arising from the distance $d_{\mathscr{G}}$ given by

$$d_{\mathscr{G}}(a,b) = (\ln a - \ln b)^2. \tag{43}$$

In fact, this should not be amazing since the arithmetic mean $\mathscr{A}$ on $\mathbb{R}$ and the geometric mean $\mathscr{G}$ on $(0,\infty)$ are *equivalent* in the sense that there is a bijection $g : (0,\infty) \to \mathbb{R}$, namely $g(x) = \ln x$, for which $\mathscr{G}(a,b) = g^{-1}\mathscr{A}(g(a),g(b))$ for all a, b. Similarly, the harmonic and arithmetic means on $(0,\infty)$ are equivalent via the bijection $h(x) = 1/x$, and therefore the harmonic mean is the distance mean arising from the distance $d_{\mathscr{H}}$ given by

$$d_{\mathscr{H}}(a,b)=\left(\frac{1}{a}-\frac{1}{b}\right)^{2}. \tag{44}$$

The analogous question pertaining to the logarithmic mean $\mathscr{L}$ defined by

$$\mathscr{L}(a,b)=\frac{a-b}{\ln a-\ln b},\quad a,b>0, \tag{45}$$

remains open.

Problem 9. Decide whether the mean $\mathscr{L}$ (defined in (45)) is a distance mean.

9. Quasi-Arithmetic Means

A k-dimensional mean $\mathscr{M}$ on $\mathbb{J}$ is called a *quasi-arithmetic* mean if there is a continuous strictly monotone function g from $\mathbb{J}$ to an interval $\mathbb{I}$ in $\mathbb{R}$ such that

$$\mathscr{M}(a_1,\dots,a_k)=g^{-1}(\mathscr{A}(g(a_1),\dots,g(a_k))) \tag{46}$$

for all a_j in $\mathbb{J}$. We have seen that the geometric and harmonic means are quasi-arithmetic and concluded that they are distance means. To see that $\mathscr{L}$ is not quasi-arithmetic, we observe that the (two-dimensional) arithmetic mean, and hence any quasi-arithmetic mean $\mathscr{M}$, satisfies the elegant functional equation

$$\mathscr{M}(\mathscr{M}(\mathscr{M}(a,b),b),\mathscr{M}(\mathscr{M}(a,b),a))=\mathscr{M}(a,b) \tag{47}$$

for all $a,b>0$. However, a quick experimentation with a random pair (a,b) shows that (47) is not satisfied by $\mathscr{L}$.

This shows that $\mathscr{L}$ is not quasi-arithmetic, but does not tell us whether $\mathscr{L}$ is a distance mean, and hence does not answer Problem 9.

The functional equation (47) is a weaker form of the functional equation

$$\mathscr{M}(\mathscr{M}(a,b),\mathscr{M}(c,d))=\mathscr{M}(\mathscr{M}(a,c),\mathscr{M}(b,d)) \tag{48}$$

for all $a,b,c,d>0$. This condition, together with the assumption that $\mathscr{M}$ is strictly increasing in each variable, characterizes two-dimensional quasi-arithmetic means; see [8, Theorem 1, pp. 287–291]. A thorough discussion of quasi-arithmetic means can be found in [3, 8].

Problem 10. Decide whether a mean $\mathscr{M}$ that satisfies the functional equation (47) (together with any necessary smoothness conditions) is necessarily a quasi-arithmetic mean.

10. Deviation Means

Deviation means were introduced in [9] and were further investigated in [10]. They are defined as follows.

A real-valued function $E=E(x,t)$ on $\mathbb{R}^2$ is called a *deviation* if $E(x,x)=0$ for all x and if $E(x,t)$ is a strictly decreasing continuous function of t for every x. If E is a deviation, and if $x_1,\dots,x_n$ are given, then the *E-deviation mean* of $x_1,\dots,x_n$ is defined to be the unique zero of

$$E(x_1,t)+\cdots+E(x_n,t). \tag{49}$$

It is direct to see that (49) has a unique zero and that this zero does indeed define a mean.

Problem 11. Characterize deviation means and explore their exact relationship with distance means.

If E is a deviation, then (following [11]), one may define d_E by

$$d_E(x,t)=\int_x^t E(x,s)\,ds. \tag{50}$$

Then $d_E(x,t)\geq 0$ and $d_E(x,t)$ is a strictly convex function in t for every x. The E-deviation mean of $x_1,\dots,x_n$ is nothing but the unique value of t at which $d_E(x_1,t)+\cdots+d_E(x_n,t)$ attains its minimum. Thus if d_E happens to be symmetric, then d_E would be a distance and the E-deviation mean would be the distance mean arising from the distance d_E.

11. Other Ways of Generating New Means

If f and g are differentiable on an open interval $\mathbb{J}$, and if $a<b$ are points in $\mathbb{J}$ such that $f(b)\neq f(a)$, then there exists, by Cauchy's mean value theorem, a point c in (a,b), such that

$$\frac{f'(c)}{g'(c)}=\frac{g(b)-g(a)}{f(b)-f(a)}. \tag{51}$$

If f and g are such that c is unique for every a, b, then we call c the *Cauchy* mean of a and b corresponding to the functions f and g, and we denote it by $\mathscr{C}_{f,g}(a,b)$.

Another natural way of defining means is to take a continuous function F that is strictly monotone on $\mathbb{J}$, and to define the mean of $a,b\in\mathbb{J}$, $a\neq b$, to be the unique point c in (a,b) such that

$$F(c)=\frac{1}{b-a}\int_a^b F(x)\,dx. \tag{52}$$

We call c the *mean value* (mean) of a and b corresponding to F, and we denote it by $\mathscr{V}(a,b)$.

Clearly, if H is an antiderivative of F, then (53) can be written as

$$H'(c)=\frac{H(b)-H(a)}{b-a}. \tag{53}$$

Thus $\mathscr{V}_F(a,b)=\mathscr{C}_{H,E}(a,b)$, where E is the identity function.

For more on the these two families of means, the reader is referred to [12] and [13], and to the references therein.

In contrast to the attitude of thinking of the mean as the number that minimizes a certain function, there is what one may call the Chisini attitude that we now describe. A function f on $\mathbb{J}^n$ may be called a *Chisini function* if and only if the equation

$$f(a_1,\dots,a_n)=f(x,\dots,x) \tag{54}$$

has a unique solution $x = a \in [a_1, a_n]$ for every ordered data set $(a_1, \ldots, a_n)$ in $\mathbb{J}$. This unique solution is called the *Chisini mean* associated to f. In Chisini's own words, *x is said to be the mean of n numbers $x_1, \ldots, x_n$ with respect to a problem, in which a function of them $f(x_1, \ldots, x_n)$ is of interest, if the function assumes the same value when all the x_h are replaced by the mean value x: $f(x_1, \ldots, x_n) = f(x, \ldots, x)$*; see [14, page 256] and [1]. Examples of such Chisini means that arise in geometric configurations can be found in [15].

Problem 12. Investigate how the families of distance, deviation, Cauchy, mean value, and Chisini means are related.

12. Internality Tests

According to the definition of a mean, all that is required of a function $\mathscr{M} : \mathbb{J}^n \to \mathbb{J}$ to be a mean is to satisfy the internality property

$$\min \{a_1, \ldots, a_k\} \le \mathscr{M} (a_1, \ldots, a_k) \le \max \{a_1, \ldots, a_k\} \quad (55)$$

for all $a_j \in \mathbb{J}$. However, one may ask whether it is sufficient, for certain types of functions $\mathscr{M}$, to verify (55) for a finite, preferably small, number of well-chosen n-tuples. This question is inspired by certain elegant theorems in the theory of copositive forms that we summarize below.

12.1. Copositivity Tests for Quadratic and Cubic Forms. By a (real) *form* in n variables, we shall always mean a homogeneous polynomial $F = F(x_1, \ldots, x_n)$ in the indeterminates $x_1, \ldots, x_n$ having coefficients in $\mathbb{R}$. When the degree t of a form F is to be emphasized, we call F a t-form. Forms of degrees 1, 2, 3, 4, and 5 are referred to as *linear*, *quadratic*, *cubic*, *quartic*, and *quintic* forms, respectively.

The set of all t-forms in n variables is a vector space (over $\mathbb{R}$) that we shall denote by $\mathbb{F}_t^{(n)}$. It may turn out to be an interesting exercise to prove that the set

$$\left\{ \prod_{j=1}^{d} N_j^{e_j} : \sum_{j=1}^{d} j e_j = d \right\} \quad (56)$$

is a basis, where N_j is the Newton polynomial defined by

$$N_j = \sum_{k=1}^{n} x_k^j. \quad (57)$$

The statement above is quite easy to prove in the special case $d \le 3$, and this is the case we are interested in in this paper. We also discard the trivial case $n = 1$ and assume always that $n \ge 2$.

Linear forms can be written as aN_1, and they are not worth much investigation. Quadratic forms can be written as

$$Q = aN_1^2 + bN_2 = a\left(\sum_{k=1}^{n} x_k\right)^2 + b\left(\sum_{k=1}^{n} x_k^2\right). \quad (58)$$

Cubic and quartic forms can be written, respectively, as

$$\begin{gathered} aN_1^3 + bN_1N_2 + cN_3, \\ aN_1^4 + bN_1^2N_2 + cN_1N_3 + dN_2^2. \end{gathered} \quad (59)$$

A form $F = F(x_1, \ldots, x_n)$ is said to be *copositive* if $f(a_1, \ldots, a_n) \ge 0$ for all $x_i \ge 0$. Copositive forms arise in the theory of inequalities and are studied in [14] (and in references therein). One of the interesting questions that one may ask about forms pertains to algorithms for deciding whether a given form is copositive. This problem, in full generality, is still open. However, for quadratic and cubic forms, we have the following satisfactory answers.

Theorem 2. *Let $F = F(x_1, \ldots, x_n)$ be a real symmetric form in any number $n \ge 2$ of variables. Let $\mathbf{v}_m^{(n)}$, $1 \le m \le n$, be the n-tuple whose first m coordinates are 1's and whose remaining coordinates are $0'$s.*

(i) *If F is quadratic, then F is copositive if and only if $F \ge 0$ at the two test n-tuples*

$$\mathbf{v}_1^{(n)} = (1, 0, \ldots, 0), \quad \mathbf{v}_n^{(n)} = (1, 1, \ldots, 1). \quad (60)$$

(ii) *If F is cubic, then F is copositive if and only if $F \ge 0$ at the n test n-tuples*

$$\mathbf{v}_m^{(n)} = \left(\overbrace{1, \ldots, 1}^{m}, \overbrace{0, \ldots, 0}^{n-m}\right), \quad 1 \le m \le n. \quad (61)$$

Part (i) is a restatement of Theorem 1(a) in [16]. Theorem 1(b) there is related and can be restated as

$$\begin{gathered} F(a_1, \ldots, a_n) \ge 0, \qquad \forall a_i \in \mathbb{R}, \\ \Longleftrightarrow F \ge 0 \text{ at the 3 } n\text{-tuples} \\ (1, 0, \ldots, 0), (1, 1, \ldots, 1), (1, -1, 0, \ldots, 0). \end{gathered} \quad (62)$$

Part (ii) was proved in [17] for $n \le 3$ and in [18] for all n. Two very short and elementary inductive proofs are given in [19].

It is worth mentioning that the n test n-tuples in (61) do not suffice for establishing the copositivity of a quartic form even when $n = 3$. An example illustrating this that uses methods from [20] can be found in [19]. However, an algorithm for deciding whether a symmetric quartic form f in n variables is copositive that consists in testing f at n-tuples of the type

$$\begin{gathered} \left(\overbrace{a, \ldots, a}^{m}, \overbrace{1, \ldots, 1}^{r}, \overbrace{0, \ldots, 0}^{n-m-r}\right), \\ 0 \le m, \ r \le n, \ m + r \le n \end{gathered} \quad (63)$$

is established in [21]. It is also proved there that if $n = 3$, then the same algorithm works for quintics but does not work for forms of higher degrees.

12.2. Internality Tests for Means Arising from Symmetric Forms. Let $\mathbb{F}_t^{(n)}$ be the vector space of all real t-forms in n variables, and let N_j, $1 \le j \le d$, be the Newton polynomials defined in (57). Means of the type

$$\mathscr{M} = \left(\frac{F_r}{F_s}\right)^{1/(r-s)}, \quad (64)$$

where F_j is a symmetric form of degree j, are clearly symmetric and 1-homogeneous, and they abound in the literature. These include the family of Gini means $G_{r,s}$ defined in (8) (and hence the Lehmer and Hölder means). They also include the elementary symmetric polynomial and elementary symmetric polynomial ratio means defined earlier in (10).

In view of Theorem 2 of the previous section, it is tempting to ask whether the internality of a function $\mathscr{M}$ of the type described in (64) can be established by testing it at a finite set of test n-tuples. Positive answers for some special cases of (64), and for other related types, are given in the following theorem.

Theorem 3. *Let L, Q, and C be real symmetric forms of degrees 1, 2, and 3, respectively, in any number $n \geq 2$ of nonnegative variables. Let $\mathbf{v}_k^{(n)}$, $1 \leq k \leq n$, be as defined in Theorem 2.*

(i) *$\sqrt{Q}$ is internal if and only if it is internal at the two test n-tuples: $\mathbf{v}_n^{(n)} = (1, 1, \ldots, 1)$ and $v_{n-1}^{(n)} = (1, 1, \ldots, 1, 0)$.*

(ii) *Q/L is internal if and only if it is internal at the two test n-tuples: $\mathbf{v}_n^{(n)} = (1, 1, \ldots, 1)$ and $v_1^{(n)} = (1, 0, \ldots, 0)$.*

(iii) *If $n \leq 4$, then $\sqrt[3]{C}$ is internal if and only if it is internal at the n test n-tuples*

$$\mathbf{v}_m^{(n)} = \Big(\overbrace{1, \ldots, 1}^{m}, \overbrace{0, \ldots, 0}^{n-m}\Big), \quad 1 \leq m \leq n. \tag{65}$$

Parts (i) and (ii) are restatements of Theorems 3 and 5 in [16]. Part (iii) is proved in [22] in a manner that leaves a lot to be desired. Besides being rather clumsy, the proof works for $n \leq 4$ only. The problem for $n \geq 5$, together with other open problems, is listed in the next problem set.

Problem Set 13. Let L, Q, and C be real symmetric cubic forms of degrees 1, 2, and 3, respectively, in n non-negative variables.

(13-a) Prove or disprove that $\sqrt[3]{C}$ is internal if and only if it is internal at the n test n-tuples

$$\mathbf{v}_m^{(n)} = \Big(\overbrace{1, \ldots, 1}^{m}, \overbrace{0, \ldots, 0}^{n-m}\Big), \quad 1 \leq m \leq n. \tag{66}$$

(13-b) Find, or prove the nonexistence of, a finite set T of test n-tuples such that the internality of C/Q at the n-tuples in T gurantees its internality at all nonnegative n-tuples.

(13-c) Find, or prove the nonexistence of, a finite set T of test n-tuples such that the internality of $L \pm \sqrt{Q}$ at the n-tuples in T guarantees its internality at all nonnegative n-tuples.

Problem (13-b) is open even for $n = 2$. In Section 6 of [15], it is shown that the two pairs $(1, 0)$ and $(1, 1)$ do not suffice as test pairs.

As for Problem (13-c), we refer the reader to [23], where means of the type $L \pm \sqrt{Q}$ were considered. It is proved in Theorem 2 there that when Q has the special form $a\prod_{1\leq i<j\leq n}(x_i - x_j)^2$, then $L \pm \sqrt{Q}$ is internal if and only if it is internal at the two test n-tuples $\mathbf{v}_n^{(n)} = (1, 1, \ldots, 1)$ and $\mathbf{v}_{n-1}^{(n)} = (1, 1, \ldots, 1, 0)$. In the general case, sufficient and necessary conditions for internality of $L \pm \sqrt{Q}$, in terms of the coefficients of L and Q, are found in [23, Theorem 3]. However, it is not obvious whether these conditions can be rewritten in terms of test n-tuples in the manner done in Theorem 3.

13. Extension of Means, Concordance of Means

The two-dimensional arithmetic mean $\mathscr{A}^{(2)}$ defined by

$$\mathscr{A}^{(2)}(a_1, a_2) = \frac{a_1 + a_2}{2} \tag{67}$$

can be extended to any dimension k by setting

$$\mathscr{A}^{(k)}(a_1, \ldots, a_k) = \frac{a_1 + \cdots + a_k}{k}. \tag{68}$$

Although very few people would disagree on this, nobody can possibly give a mathematically sound justification of the feeling that the definition in (68) is the only (or even the best) definition that makes the sequence $A^{(k)}$ of means *harmonious* or *concordant*. This does not seem to be an acceptable definition of the notion of concordance.

In a private communication several years ago, Professor Zsolt Páles told me that Kolmogorov suggested calling a sequence $\mathscr{M}^{(k)}$ of means on $\mathbb{J}$, where $\mathscr{M}^{(k)}$ is k-dimensional, concordant if for every m and n and every a_i, b_i in $\mathbb{J}$, we have

$$\begin{aligned}\mathscr{M}^{(n+m)}&(a_1, \ldots, a_n, b_1, \ldots, b_m)\\ &= \mathscr{M}^{(2)}\left(\mathscr{M}^{(n)}(a_1, \ldots, a_n), \mathscr{M}_m(b_1, \ldots, b_m)\right).\end{aligned} \tag{69}$$

He also told me that such a definition is too restrictive and seems to confirm concordance in the case of the quasi-arithmetic means only.

Problem 14. Suggest a definition of concordance, and test it on sequences of means that you feel concordant. In particular, test it on the existing generalizations, to higher dimensions, of the logarithmic mean $\mathscr{L}$ defined in (45).

14. Distance Functions in Topology

Distance functions, which are not necessarily metrics, have appeared early in the literature on topology. Given a distance function d on any set X, one may define the *open ball* $B(a, r)$ in the usual manner, and then one may declare a subset $A \subseteq X$ *open* if it contains, for every $a \in A$, an open ball $B(a, r)$ with $r > 0$. If d has the triangle inequality, then one can proceed in the usual manner to create a topology. However, for a general distance d, this need not be the case, and distances that give rise to a coherent topology in the usual manner are called *semimetrics* and they are investigated and characterized in [24–29]. Clearly, these are the distances d for which the family $\{B(a, r) : r > 0\}$ of open balls centered at $a \in S$ forms a local base at a for every a in X.

15. Centers and Center-Producing Distances

A distance d may be defined on any set S whatsoever. In particular, if d is a distance on $\mathbb{R}^2$ and if the function $f(X)$ defined by

$$f(X) = \sum_{i=1}^{n} d(X, A_i) \tag{70}$$

attains its minimum at a unique point X_0 that lies in the convex hull of $\{A_1, \ldots, A_n\}$ for every choice of $A_1, \ldots, A_n$ in $\mathbb{R}^2$, then d will be called *a center-producing distance*.

The Euclidean metric d_1 on $\mathbb{R}^2$ produces the *Fermat-Torricelli* center. This is defined to be the point whose distances from the given points have a minimal sum. Its square, d_2, which is just a distance but not a metric, produces the *centroid*. This is the center of mass of equal masses placed at the given points. It would be interesting to explore the centers defined by d_p for other values of p.

Problem 15. Let d_p, $p > 1$, be the distance defined on $\mathbb{R}^2$ by $d_p(A, B) = \|A - B\|^p$, and let ABC be a triangle. Let $Z_p = Z_p(A, B, C)$ be the point that minimizes

$$\begin{aligned} d_p(Z, A) + d_p(Z, B) + d_p(Z, C) \\ = \|Z - A\|^p + \|Z - B\|^p + \|Z - C\|^p. \end{aligned} \tag{71}$$

Investigate how Z_p, $p \geq 1$, are related to the known triangle centers, and study the curve traced by them.

The papers [30, 31] may turn out to be relevant to this problem.

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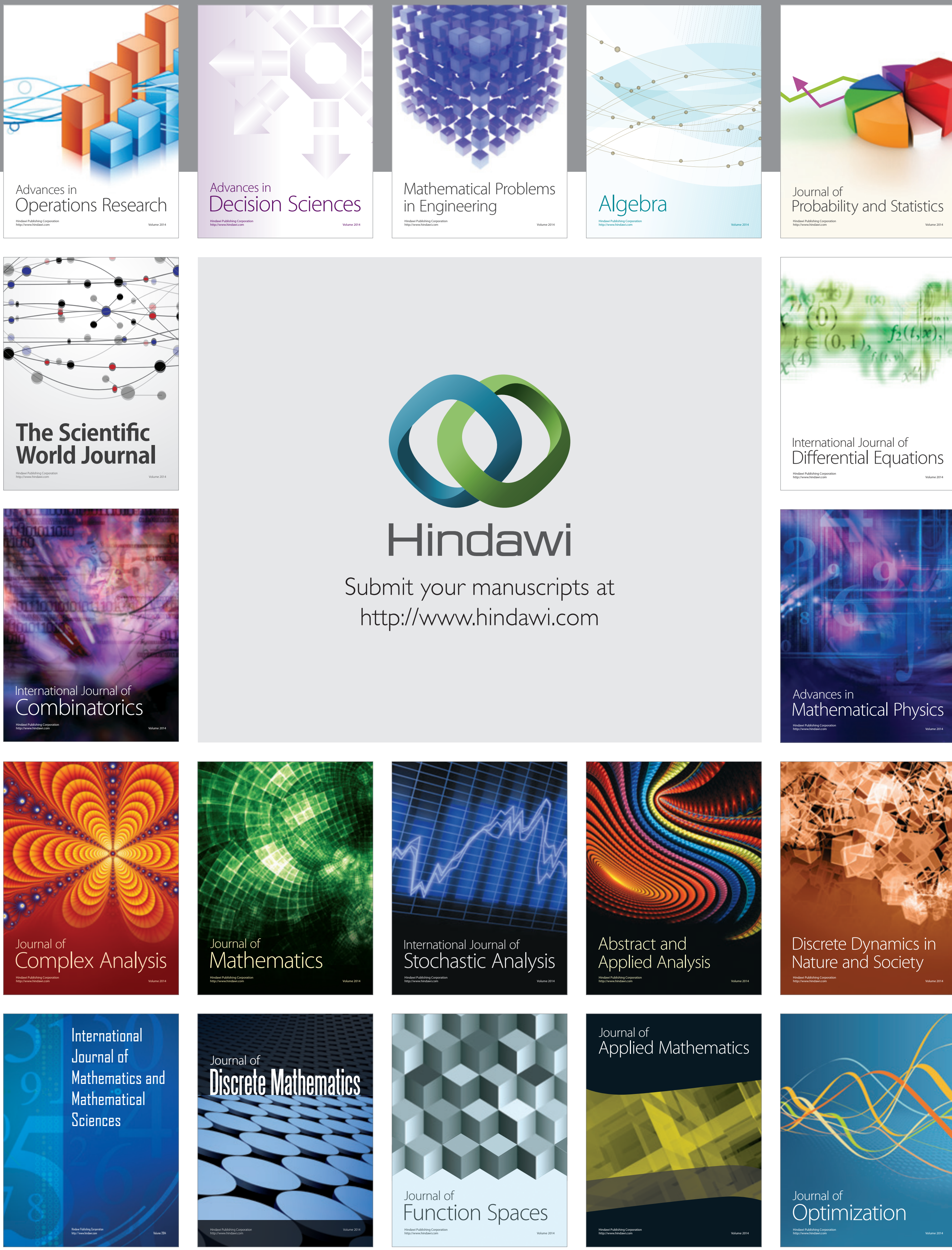
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