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ORDERED COMPACTIFICATIONS AND FAMILIES OF MAPS

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ABSTRACT. For a $T_{3.5}$-ordered space, certain families of maps are designated as "defining families." For each such defining family we construct the smallest T_2-ordered compactification such that each member of the family can be extended to the compactification space. Each defining family also generates a quasi-uniformity on the space whose bicompletion produces the same T_2-ordered 'compactification.

KEY WORDS AND PHRASES. $T_{3.5}$-ordered space, T_2-ordered compactification, defining family of maps, quasi-uniform space, bicompletion.
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INTRODUCTION.

Let X be a $T_{3.5}$-ordered space, and let $CI^*(X)$ be the set of all increasing, continuous maps from X into $[0,1]$. A subset Φ of $CI^*(X)$ which induces both the weak order and weak topology on X is called a *defining family* for X. For each such defining family Φ, we construct the smallest T_2-ordered compactification K_Φ with the property that each member of Φ can be extended to K_Φ. If Φ_1 and Φ_2 are two defining families for X such that $\Phi_1 \subseteq \Phi_2$, then $K_{\Phi_1} \leq K_{\Phi_2}$. For each defining family Φ, there is a largest defining family $\hat{\Phi}$ such that $K_\Phi = K_{\hat{\Phi}}$. Those defining families which are $\hat{\Phi}$ for some defining family Φ are called *maximal defining families*, and if Φ and Ψ are two maximal defining families, $K_\Phi \leq K_\Psi$ iff $\Phi = \Psi$. The largest defining family for X is $CI^*(X)$, and if $\Phi = CI^*(X)$ then K_Φ is the Nachbin (or Stone-Čech ordered) compactification [2].

Each defining family Φ also generates a quasi-uniformity $\mathcal{V}_\Phi$ on X (related to the "usual" quasi-uniformity $\mathcal{W}$ on $[0,1]$) which is T_0 and totally bounded. The bicompletion of $(X, \mathcal{V}_\Phi)$ (as defined in [1]) yields a uniform ordered space which, in turn, gives the compactification K_Φ. The maximal defining family $\hat{\Phi}$ is precisely the set of all quasi-uniformly continuous maps from $(X, \mathcal{V}_\Phi)$ into $([0,1], \mathcal{W})$.

1. PRELIMINARIES.

If X is a set, we denote by $\mathbf{F}(X)$ the set of all (proper) filters on X and by $\mathbf{UF}(X)$ the set of all ultrafilters on X. A non-empty collection $\mathcal{G}$ of subsets of X is called a *grill* on X if: (1) $\emptyset \notin \mathcal{G}$; (2) $A \in \mathcal{G}$ and $A \subseteq B$ implies $B \in \mathcal{G}$; (3) $A \cup B \in \mathcal{G}$ implies $A \in \mathcal{G}$ or $B \in \mathcal{G}$. With every $\mathcal{F} \in \mathbf{F}(X)$, we associate the grill $\gamma(\mathcal{F}) = \{A \subseteq X : X \setminus A \notin \mathcal{F}\}$; equivalently, $\gamma(\mathcal{F})$ is the union of all ultrafilters finer than $\mathcal{F}$.

Let $(X, \leq)$ be a poset; A subset $A \subseteq X$ is *increasing* (respectively, *decreasing*) if $x \in A$ and $x \leq y$ (respectively, $y \leq x$) implies $y \in A$. If $(X, \leq)$ and $(Y, \leq^*)$ are posets, then a mapping f :

$(X, \leq) \to (Y, \leq^*)$ is *increasing* (respectively, decreasing) if $x \leq y$ implies $f(x) \leq^* f(y)$ (respectively, $f(y) \leq^* f(x)$).

An ordered space $(X, \tau, \leq)$ consists of a poset $(X, \leq)$ and a topology τ on $(X, \leq)$ which is *convex* (meaning that the collection of all τ-open sets which are either increasing or decreasing is a subbase for τ). Usually an ordered space $(X, \tau, \leq)$ will simply be denoted by X. The closed unit interval $[0, 1]$ with its usual order and topology is designated by I. For any ordered space X, let $CI^*(X)$ (respectively, $CD^*(X)$) be the set of all continuous increasing (respectively, decreasing) maps from X into I. More generally, for ordered spaces X and Y, $CI(X, Y)$ represents the set of all continuous, increasing functions from X into Y.

An ordered space X is said to be *T_2-ordered* if the order "$\leq$" is closed in $X \times X$. A T_2-ordered space X which has both the weak order (see Condition (W_o) below) and weak topology induced by $CI^*(X)$ is said to be *$T_{3.5}$-ordered* (or *completely regular ordered* in the terminology of [2]). Some well-known characterizations of $T_{3.5}$-ordered spaces are summarized in the following proposition.

PROPOSITION 1.1 The following statements about an ordered space X are equivalent.

(1) X is $T_{3.5}$-ordered.

(2) X is a subspace of a compact, T_2-ordered space.

(3) X satisfies the following conditions:

(i) If $x \in X$, A is a closed subset of X, and $x \notin A$, then there is $f \in CI^*(X)$ and $g \in CD^*(X)$ such that $f(x) = g(x) = 0$ and $f(y) \vee g(y) = 1$, for all $y \in A$;

(ii) If $x \not\leq y$ in X, there is $f \in CI^*(X)$ such that $f(y) = 0$ and $f(x) = 1$.

(4) The order and topology for X are induced by some quasi-uniformity $\mathcal{W}$ on X (i.e., $\cap\mathcal{W}$ is the order for X and the topology of X is the uniform topology of the uniformity $\mathcal{W} \vee \mathcal{W}^{-1}$).

Every $T_{3.5}$-ordered space X has a largest T_2-ordered compactification $\beta_o X$ called the Nachbin compactification, which can be constructed by embedding X in the "ordered cube" $I^{CI^*(X)}$, with the product order and topology.

Let X be an ordered space. If Φ is any subset of $CI^*(X)$ such that X has the weak order and the weak topology determined by Φ, then Φ is called a *defining family* for X. More precisely, $\Phi \subseteq CI^*(X)$ is a defining family if the following conditions are satisfied:

(W_τ) For any $\mathcal{F} \in \mathbf{UF}(X)$, $\mathcal{F} \to x$ in X iff $f(\mathcal{F}) \to f(x)$ in I, for all $f \in \Phi$.

(W_o) For any $(x, y) \in X \times X$, $x \leq y$ in X iff $f(x) \leq f(y)$ in I, for all $f \in \Phi$.

Some rather obvious remarks about defining families are summarized in the next proposition.

PROPOSITION 1.2 Let X be an ordered space.

(1) X is $T_{3.5}$-ordered iff X allows at least one defining family. In particular, $CI^*(X)$ is a defining family for every $T_{3.5}$-ordered space.

(2) If $\Phi_1 \subseteq \Phi_2 \subseteq CI^*(X)$ and Φ_1 is a defining family for X, then Φ_2 is also a defining family for X.

2. THE COMPACTIFICATION K_Φ.

Let X be a $T_{3.5}$-ordered space. If $\mathcal{F} \in \mathbf{UF}(X)$ and $f \in CI^*(X)$, there is a unique point $a_{\mathcal{F},f}$ in I such that $f(\mathcal{F}) \to a_{\mathcal{F},f}$. For any $a \in I$, let $\mathcal{V}(a)$ denote the neighborhood filter at a. If Φ is a defining family for X and $\mathcal{F} \in \mathbf{UF}(X)$, we define the filter $\mathcal{F}_\Phi = \vee\{f^{-1}(\mathcal{V}(a_{\mathcal{F},f})) : f \in \Phi\}$. Note that if $\mathcal{F} \to x$ in X, then $a_{\mathcal{F},f} = f(x)$ for all $f \in \Phi$, and in this case $\mathcal{F}_\Phi$ is simply the neighborhood filter at x.

Continuing with the assumptions of the preceding paragraph, let $X_\Phi = \{\gamma(\mathcal{F}_\Phi) : \mathcal{F} \in \mathbf{UF}(X)\}$ be the set of grills associated with the filters $\mathcal{F}_\Phi$. If $\gamma \in X_\Phi$ and $\mathcal{F}, \mathcal{G} \in \mathbf{UF}(X)$ are such that $\mathcal{F} \subseteq \gamma$ and $\mathcal{G} \subseteq \gamma$, then $a_{\mathcal{F},f} = a_{\mathcal{G},f}$, for all $f \in \Phi$. It therefore follows that, for each $f \in \Phi$, the function

$f_\Phi : X_\Phi \to I$, defined by $f_\Phi(\gamma) = a_{\mathcal{F},f}$, where $\mathcal{F}$ any ultrafilter that is a subset of γ, is well-defined. If $i_\Phi : X \to X_\Phi$ is defined by $i_\Phi(x) = \gamma(\dot{x}_\Phi)$, where $\dot{x}$ is the fixed ultrafilter generated by $\{x\}$, then clearly i_Φ is an injection and the diagram below commutes for every $f \in \Phi$.

$$\begin{tikzcd} X_\Phi \arrow[dr, "f_\Phi"] & \\ & I \\ X \arrow[uu, "i_\Phi"] \arrow[ur, "f"'] & \end{tikzcd}$$

Let X_Φ be equipped with the weak order and weak topology induced by $\{f_\Phi : f \in \Phi\}$. Then i_Φ is an *ordered space embedding* (i.e., i_Φ is topological embedding, and $x \leq y \Leftrightarrow i_\Phi(x) \leq_\Phi i_\Phi(y)$, where $\leq_\Phi$ denotes the order of X_Φ).

THEOREM 2.1 Let X be a $T_{3.5}$-ordered space and Φ a defining family for X. Then (X_Φ, i_Φ) is a T_2-ordered compactification of X, and each $f \in \Phi$ has a unique, continuous, increasing extension to X_Φ such that the diagram below commutes.

$$\begin{tikzcd} X_\Phi \arrow[dr, "f_\Phi"] & \\ & I \\ X \arrow[uu, "i_\Phi"] \arrow[ur, "f"'] & \end{tikzcd}$$

PROOF. The family $\Phi^\wedge = \{f_\Phi : f \in \Phi\}$ separates points in X_Φ, and therefore X_Φ is $T_{3.5}$-ordered; in particular, X_Φ is T_2-ordered. In view of the paragraph preceding the theorem, it remains only to show that X_Φ is compact and $i_\Phi(X)$ is dense in X_Φ.

Let $\mathcal{A} \in \mathbf{UF}(X)$. For each $\gamma \in X_\Phi$, choose an ultrafilter $\mathcal{F}_\gamma$ such that $\mathcal{F}_\gamma \subseteq \gamma$; in particular, if $\gamma = \gamma(\mathcal{F}_\Phi)$ where $\mathcal{F} \to x$ in X, define $\mathcal{F}_\gamma = \dot{x}$. If $B \subseteq X$, let $B^* = \{\gamma \in X_\Phi : B \in \mathcal{F}_\gamma\}$. Then, define $\mathcal{F}_\mathcal{A} = \{A \subseteq X : A^* \in \mathcal{A}\}$; one easily verifies that $\mathcal{F}_\mathcal{A}$ is an ultrafilter. We shall show that $\mathcal{A} \to \gamma(\mathcal{F}_\mathcal{A})$ in X_Φ. For this purpose, it suffices to show that $f_\Phi(\mathcal{A}) \to f_\Phi(\gamma(\mathcal{F}_\mathcal{A})) = a_{\mathcal{F}_\mathcal{A},f}$, for all $f \in \Phi$. Given $f \in \Phi$, let U be a closed neighborhood of $a_{\mathcal{F}_\mathcal{A},f}$ in I. We first observe that $f(\mathcal{F}_\mathcal{A}) \to a_{\mathcal{F}_\mathcal{A},f}$, and hence $f^{-1}(U) \in \mathcal{F}_\mathcal{A}$, which implies $(f^{-1}(U))^* \in \mathcal{A}$. Then note that $f(\mathcal{F}_\mathcal{A}) \to a_{\mathcal{F}_\mathcal{A},f}$; consequently $f_\Phi^{-1}(U) \in \mathcal{A}$, and $f_\Phi(\mathcal{A}) \to a_{\mathcal{F}_\mathcal{A},f}$. Thus X_Φ is compact.

Finally, let $\gamma \in X_\Phi$ and, for $B \subseteq X$, let B^* be defined as in the preceding paragraph. If $\mathcal{F} \in \mathbf{UF}(X)$ and $\mathcal{F} \subseteq \gamma$, let $\mathcal{F}^*$ be the filter on X_Φ generated by $\{F^* : F \in \mathcal{F}\}$. One easily shows that $\mathcal{F}^* \to \gamma$ in X_Φ. Since $i_\Phi(\mathcal{F}) \geq \mathcal{F}^*$, it follows that $i_\Phi(X)$ is dense in X_Φ. ∎

The compactification (X_Φ, i_Φ) of X determined by a defining family Φ will be denoted by K_Φ. By the preceding theorem, each $f \in \Phi$ has a unique extension $f_\Phi \in CI^*(X_\Phi)$. If Y is any compact, T_2-ordered space, we define $CI_\Phi(X,Y) = \{f \in CI(X,Y) : h \circ f \in \Phi$, for all $h \in CI^*(Y)\}$. The next theorem establishes that each $f \in CI_\Phi(X,Y)$ can be "lifted" relative to K_Φ.

THEOREM 2.2 Let X be a $T_{3.5}$-ordered space, Φ a defining family for X, and Y a compact, T_2-ordered space. If $g \in CI_\Phi(X,Y)$, then there is a unique $g_\Phi \in CI(X_\Phi, Y)$ such that the diagram below commutes.

$$\begin{tikzcd} X_\Phi \arrow[dr, "g_\Phi"] & \\ & Y \\ X \arrow[uu, "i_\Phi"] \arrow[ur, "g"'] & \end{tikzcd}$$

PROOF. Let $g \in CI_\Phi(X,Y)$ and $\gamma \in X_\Phi$; assume $\mathcal{F}$ is an ultrafilter and $\mathcal{F} \subseteq \gamma$. Define $g_\Phi : X_\Phi \to Y$ as following: $g_\Phi(\gamma) = y_{\mathcal{F},g}$, where $y_{\mathcal{F},g}$ is the unique limit of $g(\mathcal{F})$ in Y. Using the facts that $g \in CI_\Phi(X,Y)$ and $CI^*(Y)$ separates points in Y, we see that $g_\Phi(\gamma)$ is independent of the ultrafilter $\mathcal{F}$ which represents γ, so g_Φ is well defined.

If $h \in CI^*(Y)$, let $h' = h \circ g$. Then we observe that the preceding definition of g_Φ makes the following diagram commutes:

$$\begin{tikzcd}
X_\Phi \arrow[dr, "h'_\Phi"] \arrow[drr, "g_\Phi"] & & \\
 & I & Y \arrow[l, "h"'] \\
X \arrow[uu, "i_\Phi"] \arrow[ur, "h'"] \arrow[urr, "g"'] & &
\end{tikzcd}$$

If $\gamma \leq \delta$ in X_Φ, then $h'_\Phi(\gamma) \leq h'_\Phi(\delta), \forall h \in CI^*(Y)$, which implies $h(g_\Phi(\gamma)) \leq h(g_\Phi(\delta))$ holds for all $h \in CI^*(Y)$. Since Y has the weak order induced by $CI^*(Y)$, $g_\Phi(\gamma) \leq g_\Phi(\delta)$. Thus g_Φ is increasing. A similar argument, based on Y having the weak topology induced by $CI^*(Y)$, shows that g_Φ is continuous. The uniqueness of g_Φ is obvious because all spaces involved are Hausdorff. ∎

We omit the simple proof of the next proposition.

PROPOSITION 2.3 If Φ is a defining family for a $T_{3.5}$-ordered space X, then $\Phi' = \{f_\Phi : f \in CI^*(X)\}$ is a defining family for X_Φ.

Starting with a $T_{3.5}$-ordered space X and a defining family Φ for X, it follows that Φ' and $CI^*(X_\Phi)$ are both defining families for X_Φ, and it is clear that $\Phi' \subseteq CI^*(X_\Phi)$. Let $\hat{\Phi} = \{f \in CI^*(X) : \text{there is } g \in CI^*(X_\Phi) \text{ such that } f = g \circ i_\Phi\}$; in other words, $\hat{\Phi}$ consists of all members of $CI^*(X)$ which have a continuous, increasing extension in $CI^*(X_\Phi)$. Clearly $\Phi \subseteq \hat{\Phi}$, and so $\hat{\Phi}$ is a defining family for X. Note that $(\hat{\Phi})' = CI^*(X_\Phi)$, and since $(\hat{\Phi})'$ is, by Proposition 2.3, a defining family for $X_{\hat{\Phi}}$, it follows that $X_{\hat{\Phi}} = X_\Phi$. These observations yield the following result.

PROPOSITION 2.4 If Φ is a defining family for a $T_{3.5}$-ordered space X, then $\hat{\Phi} = \{f \in CI^*(X) : \text{there is } g \in CI^*(X_\Phi) \text{ such that } f = g \circ i_\Phi\}$ is the largest defining family for X such that $K_\Phi = K_{\hat{\Phi}}$.

THEOREM 2.5 Let Φ, Ψ be defining families for a $T_{3.5}$-ordered space.
(a) If $\Phi \subseteq \Psi$, then $K_\Phi \leq K_\Psi$.
(b) $K_\Phi \leq K_\Psi$ iff $\hat{\Phi} \subseteq \hat{\Psi}$.

PROOF. (a) $\Phi \subseteq \Psi$ implies $\hat{\Phi} \subseteq \hat{\Psi}$. Considering the diagram

$$\begin{tikzcd}
 & X_{\hat{\Psi}} \arrow[dd, "(i_{\hat{\Phi}})_{\hat{\Psi}}"] \\
X \arrow[ur, "i_{\hat{\Psi}}"] \arrow[dr, "i_{\hat{\Phi}}"'] & \\
 & X_{\hat{\Phi}}
\end{tikzcd}$$

and applying Theorem 2.2, we see that $(i_{\hat{\Phi}})_{\hat{\Psi}}$ is increasing and continous. Thus $K_\Phi = K_{\hat{\Phi}} \leq K_{\hat{\Psi}} = K_\Psi$.

(b) If $K_\Phi \leq K_\Psi$, then there is an increasing, continous map σ making the diagram

$$\begin{tikzcd}
 & X_\Psi \arrow[dd, "\sigma"] \\
X \arrow[ur, "i_\Psi"] \arrow[dr, "i_\Phi"'] & \\
 & X_\Phi
\end{tikzcd}$$

commute. Each member of $\hat{\Phi}$ has the form $f \circ i_\Phi$ for some $f \in CI^*(X_\Phi)$. But $f \circ i_\Phi = f \circ \sigma \circ i_\Psi$ is also in $\hat{\Psi}$. Thus $\hat{\Phi} \subseteq \hat{\Psi}$. The converse follows from (a). ∎

If X is a $T_{3.5}$-ordered space, let $\mathbf{DF}(X)$ be the poset of all defining families, ordered by inclusion. Two defining families Φ and Ψ in $\mathbf{DF}(X)$ are *equivalent* if $K_\Phi = K_\Psi$ (i.e., if K_Φ and K_Ψ are equivalent compactifications of X in the usual sense). Thus $\mathbf{DF}(X)$ is partitioned into equivalent classes, and each equivalent class $\langle\Phi\rangle$ contains a largest member $\hat{\Phi}$ which we call a *maximal defining family.*

COROLLARY 2.6 Let X be a $T_{3.5}$-ordered space, $K = (X', \phi)$ a T_2-ordered compactification of X, and $\Phi \in \mathbf{DF}(X)$ such that each $f \in \Phi$ has an extension $f' \in CI^*(X')$. Then $K_\Phi \leq K$.

COROLLARY 2.7 For a $T_{3.5}$-ordered space, the correspondence $\Phi \longleftrightarrow K_\Phi$ is bijective and order-preserving between the maximal defining families for X and the T_2-ordered compactifications of X.

3. DEFINING FAMILIES AND QUASI-UNIFORMITIES.

This concluding section is based on the results of Fletcher and Lindgren [1], and to some extent we borrow their notation.

Let $(X, \mathcal{V})$ be a quasi-uniform space; the associated uniformity $\mathcal{V} \vee \mathcal{V}^{-1}$ will be denoted by $\mathcal{V}^*$. Recall that $(X, \mathcal{V})$ is T_0 iff $\cap\mathcal{V}$ is a partial order (or, equivalently, $(X, \mathcal{V}^*)$ is T_2), and *totally bounded* iff, for each $U \in \mathcal{V}$, there is a finite cover $\{A_1, \cdots, A_n\}$ of X such that $A_i \times A_i \subseteq U$, for $i = 1, \cdots, n$. Note that $\mathcal{V}$ is totally bounded iff $\mathcal{V}^*$ is totally bounded.

Every T_0, quasi-uniform space $(X, \mathcal{V})$ induces a *uniform ordered space* $(X, \mathcal{U}, \leq)$, where $\mathcal{U} = \mathcal{V}^*$ and "$\leq$"$=\cap\mathcal{V}$; also associated with $(X, \mathcal{V})$ is the $T_{3.5}$-ordered space $(X, \tau, \leq)$, where $\tau = \tau_{\mathcal{V}^*}$ and "$\leq$" is again $\cap\mathcal{V}$. Furthermore, for every compact, T_2-ordered space $(X, \tau, \leq)$, there is a unique quasi-uniformity $\mathcal{V}$ on X such that $\tau = \tau_{\mathcal{V}^*}$ and "$\leq$" $= \cap\mathcal{V}$ (Theorem 4.21, [1]). In particular, for the compact, T_2-ordered space I, the unique compatible quasi-uniformity, denoted here by $\mathcal{W}$, has a base of sets of the form $W_\epsilon = \{(x, y) \in I \times I : x - y \leq \epsilon\}$, where $\epsilon > 0$.

For a quasi-uniform space $(X, \mathcal{V})$, let $QUC(X, \mathcal{V})$ be the set of all quasi-uniformly continous maps from $(X, \mathcal{V})$ into $(I, \mathcal{W})$. If $X = (X, \tau, \leq)$ is the $T_{3.5}$-ordered space associated with $(X, \mathcal{V})$, it is clear that $QUC(X, \mathcal{V}) \subseteq CI^*(X)$. It is shown in Theorems 3.29 and 3.33 of [1] that every T_0, quasi-uniform space $(X, \mathcal{V})$ has a *bicompletion* $((\tilde{X}, \tilde{\mathcal{V}}), \jmath)$ such that $((\tilde{X}, (\tilde{\mathcal{V}})^*), \jmath)$ is the unique uniform space completion of $(X, \mathcal{V}^*)$, and each $f \in QUC(X, \mathcal{V})$ has a unique extension in $QUC(\tilde{X}, \tilde{\mathcal{V}})$. These observations lead to the following proposition.

PROPOSITION 3.1 Let $(X, \mathcal{V})$ be a T_0, totally bounded quasi-uniform space with associated $T_{3.5}$-ordered space $(X, \tau, \leq)$, and let $((\tilde{X}, \tilde{\mathcal{V}}), \jmath)$ be the bicompletion of $(X, \mathcal{V})$. If $(\tilde{X}, \tilde{\tau}, \tilde{\leq})$ is the $T_{3.5}$-ordered space associated with $(\tilde{X}, \tilde{\mathcal{V}})$, then $\tilde{K} = ((\tilde{X}, \tilde{\tau}, \tilde{\leq}), \jmath)$ is a T_2-ordered compactification of $(X, \tau, \leq)$.

THEOREM 3.2 Let X be a $T_{3.5}$-ordered space and $\Phi \in \mathbf{DF}(X)$. Let $\mathcal{V}_\Phi$ be the weak uniformity on X induced by Φ relative to $(I, \mathcal{W})$. Let $((\tilde{X}_\Phi, \tilde{\mathcal{V}}_\Phi), \jmath)$ be the bicompletion of $(X, \mathcal{V}_\Phi)$, and $\tilde{K}_\Phi = ((\tilde{X}_\Phi, \tilde{\tau}_\Phi, \tilde{\leq}), \jmath)$ be the T_2-ordered compactification of X induced by the bicompletion. Then $\tilde{K}_\Phi = K_\Phi$.

PROOF. Let $\mathcal{V}$ be the unique, T_0 totally bounded quasi-uniformity on X_Φ whose associated $T_{3.5}$-ordered space is the compactification $((X_\Phi, \tau_\Phi, \leq_\Phi), i_\Phi)$ derived from Φ. The latter space has the weak order and topology induced by Φ' (see Proposition 2.3) relative to I, and hence $\mathcal{V}$ is the weak quasi-uniformity on X_Φ induced by Φ' relative to $(I, \mathcal{W})$. If $\mathcal{U} = (i_\Phi)^{-1}(\mathcal{V})$ is the restriction of $\mathcal{V}$ to X, then $\mathcal{U}$ is the weak quasi-uniformity on X induced by Φ relative to $(I, \mathcal{W})$. In other words, $\mathcal{U} = \mathcal{V}_\Phi$. Since the T_2-ordered compactification associated with a T_0, totally bounded quasi-uniformity is unique (up to equivalence), $\tilde{K}_\Phi = K_\Phi$. ∎

COROLLARY 3.3 Let X be a $T_{3.5}$-ordered space and $\Phi \in \mathbf{DF}(X)$. Then $\hat{\Phi} = QUC(X, \mathcal{V}_\Phi)$.

PROOF. By Theorem 3.29, [1], each $f \in QUC(X, \mathcal{V}_\Phi)$ can be extended to the compactification $\tilde{K}_\Phi = K_\Phi$; thus $QUC(X, \mathcal{V}_\Phi) \subseteq \hat{\Phi}$. Conversely, each $f \in \hat{\Phi}$ has a unique, increasing, continuous extension to $K_\Phi = \tilde{K}_\Phi$, and this extension of f is quasi-uniformly continuous from $(\tilde{X}_\Phi, \tilde{\mathcal{V}}_\Phi)$ into

$(I, \mathcal{W})$. Thus $f \in QUC(X, \mathcal{V}_\Phi)$. ∎

COROLLARY 3.4 Let $(X, \mathcal{V})$ be a T_0, totally bounded quasi-uniform space with associated compact, T_2-ordered space $X = (X, \tau, \leq)$. Then $\Phi = QUC(X, \mathcal{V})$ is a maximal defining family for X and $\mathcal{V} = \mathcal{V}_\Phi$.

COROLLARY 3.5 Let X be a $T_{3.5}$-ordered space. Then $\mathcal{V} \longleftrightarrow QUC(X, \mathcal{V})$ is bijective and order-preserving between the T_0, totally bounded quasi-uniformities which induce X and the maximal defining families for X.

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