Internat. J. Math. & Math. Sci.
VOL. 20 NO. 3 (1997) 529-538

PURE BAER INJECTIVE MODULES

NADA M. AL THANI
Department of Mathematics
Faculty of Science
Qatar University
Doha, QATAR

(Received April 19,1995 and in revised form April 16, 1996)

ABSTRACT. In this paper we generalize the notion of pure injectivity of modules by introducing what we call a pure Baer injective module. Some properties and some characterization of such modules are established. We also introduce two notions closely related to pure Baer injectivity; namely, the notions of a Σ-pure Baer injective module and that of SSBI-ring. A ring R is an SSBI-ring if and only if every semisimple R-module is pure Baer injective. To investigate such algebraic structures we had to define what we call p-essential extension modules, pure relative complement submodules, left pure hereditary rings and some other related notions. The basic properties of these concepts and their interrelationships are explored, and are further related to the notions of pure split modules.

KEY WORDS AND PHRASES: Pure and Σ-pure Baer injective modules, pure hereditary ring, pure-split module, P-essential extension submodule, pure relative complement submodule, SSBI-ring.
1991 AMS SUBJECT CLASSIFICATION CODES: 13C.

0. INTRODUCTION

In this introduction we establish terminology and recall a summary of some basic definitions and results in the literature necessary for subsequent sections of the paper.

Throughout R will denote a ring with identity. Unless otherwise stated, all modules will be unitary left R-modules, and all homomorphisms will be R-homomorphisms.

A short exact sequence $0 \to N \to M \to K \to 0$ of left R-modules is said to be pure if $L \otimes_R N \to L \otimes_R M$ is a monomorphism for every right R-module L. A submodule N of an R-module M is called a pure submodule of M in case the natural homomorphism $L \otimes_R N \to L \otimes_R M$ is injective for every right R-module L. Equivalently, N is pure in M if and only if for any finite system of linear equations $\sum_{j=1}^{n} r_{ij} x_j = a_i$, $1 \leq i \leq m$, where $r_{ij} \in R$ and $a_i \in N$, if the system has a solution $(s_1, ..., s_n) \in M^n$, it also has a solution $(t_1, ..., t_n) \in N^n$.

A left R-module is regular if and only if every submodule of M is pure, and a submodule N of a flat R-module M is pure if $IN = IM \cap N$ for all right ideals I of R. For further results concerning this type of purity see [1] and [2].

A left ideal I of a ring R is pure in R if and only if for every $x \in I$ there exists an $y \in I$ such that $x = xy$ Furthermore, a ring R is Von-Neumann regular if and only if each left (right) R-module is flat if and only if every (principal) left (right) ideal is pure (see [3]).

A submodule N of an R-module M is called relatively divisible or briefly RD-submodule if $rN = N \cap rM$ for all $r \in R$. A commutative domain R is Prüfer if and only if every finitely generated

ideal is projective. For modules over a Prüfer ring, purity and RD-purity coincide, flatness and torsion freeness coincide (see [4]).

By $K \trianglelefteq M$ we shall understand that K is an essential submodule of M.

A ring R is called a left V-ring if and only if every simple left R-module is injective. Kaplansky proved that a commutative ring R is a V-ring if and only if it is regular [5].

A left R-module M is called pure injective if it is injective relative to every pure exact sequence of R-modules Warfield [4] proved that any left R-module can be embedded as a pure submodule of a pure injective R-module. It can also be proved that a pure injective R-module is a direct summand of every R-module containing it as a pure submodule (see [6]).

For further related results we refer to [2], [3] and [4], together with the monographs [1], [6] and [7]

1. PURE BAER INJECTIVE MODULES

We now introduce the definition of a pure Baer injective module

DEFINITION 1.1. An R-module M is called a pure Baer injective module if for each pure left ideal I of R, any R-homomorphism $f : I \to M$ can be extended to an R-homomorphism $\overline{f} : R \to M$.

If a ring R is free from non-zero one sided zero divisors then any R-module is necessarily pure Baer injective. In fact R does not possess any non-zero proper pure one-sided ideal in this case. This means, in particular, that any abelian group is a pure Baer injective $\mathbb{Z}$-module.

We find it necessary to point out from the start that the notion of pure Baer injectivity is different from that of pure injectivity; as an example consider the $\mathbb{Z}$-module $\mathbb{Z}$. However, it is evident that every pure injective R-module is pure Baer injective. Furthermore, we note the easily deduced fact that any pure Baer injective module over a Von-Neumann regular ring is injective.†

The following result is essential in characterizing pure semisimplicity of rings, a notion to be introduced in the sequel.

THEOREM 1.2 (Pure Baer Injectivity Test). For a left R-module the following are equivalent:

(1) M is pure Baer injective R-module;

(2) For every pure left ideal I of R and every R-homomorphism $f : I \to M$, there exists an $m \in M$ such that for all $a \in I$, $f(a) = am$;

(3)† For every pure exact sequence

$$0 \to I \to R \to R/I \to 0$$

the sequence

$$M \to Hom_R(I, M) \to 0$$

is exact.

PROOF. Clear. □

PROPOSITION 1.3. The direct product $\Pi_i M_i$ of R-modules is pure Baer injective if and only if each M_i is pure Baer injective.

PROOF. Clear. □

We recall that a module M over a ring R is called torsion-free if for no $0 \neq r \in R$, $rx = 0$ unless $0 = x \in M$.

The proof of the previous result shows that if $\oplus_i M_i$ is pure Baer injective, then so is each M_i However, we have:

PROPOSITION 1.4. A direct sum $\oplus_\alpha M_\alpha$ of torsion-free R-modules is pure Baer injective if and only if each M_α is pure Baer injective.

PROOF. Let M_α be pure Baer injective for each α and consider any R-homomorphism $f : I \to \oplus_\alpha M_\alpha$, I being a pure left ideal of R. Considering the canonical projection π_α on M_α and

†I wish to thank my study supervisors for calling my attention to this result.

applying the pure Baer injectivity test Lemma 1.2, one can find $m_\alpha \in M_\alpha$ for each α such that $\pi_\alpha f(x) = xm_\alpha$ for every $x \in I$. Thus $f(x) = \sum_{\text{finite}} xm_\alpha$. This means that $xm_\alpha = 0$ for all but a finite number of the indices. But since each M_α is torsion-free, we conclude that $m_\alpha = 0$ for almost all α This means that the element $\{m_\alpha\}$ of $\Pi_\alpha M_\alpha$ is an element of $\oplus_\alpha M_\alpha$, proving that this direct sum is pure Baer injective □

2. THE NOTIONS OF PURE HEREDITARY AND PURE SIMPLE RINGS AND THEIR ROLE IN PURE BAER INJECTIVE MODULES

In this section we define and study two notions: the pure hereditary ring and the pure simple ring, both of which appear to have a vital role in characterizing pure Baer modules.

DEFINITION 2.1. A ring R is called left pure hereditary if every pure left ideal of R is projective.

In what follows we shall prove that the class of pure Baer injective R-modules over a pure hereditary ring, is homomorphically closed.

THEOREM 2.2. The following statements are pair-wise equivalent for a given ring R

(1) R is left pure hereditary;

(2) The homomorphic image of a pure Baer injective R-module is pure Baer injective;

(3) The homomorphic image of an injective R-module is pure Baer injective;

(4) Any finite sum of injective submodules of an R-module is pure Baer injective.

PROOF. (1) ⇒ (2) Consider the following diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & I & \longrightarrow & R \\ & & f \downarrow & & \\ M & \xrightarrow{g} & K & \longrightarrow & 0 \end{array}$$

of R-modules, where I is a pure left ideal of R, and M is pure Baer injective. Projectivity of I shows that for some R-homomorphism $\varphi : I \to M$, $f = g\varphi$. Moreover there exists a homomorphism $\overline{\varphi} : R \to M$ that extends φ. This shows that $\overline{f} = g\overline{\varphi}$ is an extension of f; and so K is pure Baer injective as required.

(2) ⇒ (3) Clear.

(3) ⇒ (1) Let I be a pure left ideal of R and M be a left R-module whose injective hull is $E(M)$ Consider the following diagram of R-homomorphisms

$$\begin{array}{ccccc} 0 & \longrightarrow & I & \xrightarrow{i} & R \\ & & f \downarrow & & \\ E(M) & \xrightarrow{g} & K & \longrightarrow & 0 \end{array}$$

recalling that K is pure Baer injective by assumption. So, there exists an R-homomorphism $h : R \to K$ whose restriction on I is f. Again since R is projective, there exists an R-homomorphism $\sigma : R \to E(M)$ such that $g\sigma = h$; and so $g\sigma i = f$. This means that I is $E(M)$-projective Thus I is M-projective (for the proof cf. [5, p. 180, prop 16.12]); and so I is projective

(3) ⇒ (4) Clear

(4) ⇒ (3) Let N be a submodule of an injective R-module E. To prove that E/N is pure Baer injective we consider the submodule $K = \{(x, x) : x \in N\}$ of $Q = E \oplus E$ and the two submodules

$M_1 = \{(x,0)+K : x \in E\}$ and $M_2 = \{(0,y)+K : y \in E\}$ of Q/K. Now $Q/K = M_1 + M_2$, also, $M_1 \cap M_2 = \{(x,0)+K : x \in N\}$. For if $x \in N$, then $(x,0)+K = (0,-x)+K \in M_1 \cap M_2$; on the other hand the assumption that $(x,0)+K = (0,y)+K$ where $x, y \in E$ means that $x = -y \in N$ and therefore $M_1 \cap M_2 = \{(x,0)+K : x \in N\}$ Define $f : E \to M_1$ with $f(x) = (x,0)+K$ for all $x \in E$ to obtain the isomorphisms $E \cong M_1$ and $N \cong M_1 \cap M_2$. Similarly $E \cong M_2$. Thus $Q/K = M_1 + M_2$ is pure Baer injective. This shows that for some R-module G, $Q/K = M_1 \oplus G$ and $G \cong (Q/K)/M_1 \cong M_2/M_1 \cap M_2 \cong E/N$. Now since G is pure Baer injective, E/N is pure Baer injective as to be proved. □

If R is left self-injective, the pure Baer injectivity of each homomorphic image ${}_RR$ can be discussed in view of the following.

PROPOSITION 2.3. Let R be a left self-injective ring. If R/J is pure Baer injective for each essential left ideal J, then R/I is pure Baer injective for every left ideal I of R.

PROOF. Let I be a left ideal of R and let $E(I)$ be its injective hull. Now, since $E(I)$ is a direct summand of R, there exists an idempotent $e \in E(I)$ such that $E(I) = Re$. Consider the R-homomorphism $f : R \to re$, with $f(r) = re$ for each $r \in R$. Since $I \trianglelefteq Re$, $f^{-1}(I) \trianglelefteq R$. Therefore by hypothesis $R/f^{-1}(I)$ is pure Baer injective. Define $\overline{f} : Re/I \to R/f^{-1}(I)$ by $\overline{f}(re+I) = r + f^{-1}(I)$ This is a well-defined R-isomorphism; and so Re/I is pure Baer injective. We prove that the sum $B = R(1-e)+I$ is a direct sum. If $x \in R(1-e) \cap I$, then $x = r - re = r'e$ for some $r, r' \in R$ and so $x = 0$. Thus $B/I \cong R(1-e)$ showing the pure Baer injectivity of B/I. Furthermore we have $R/I = B/I \oplus Re/I$. To see this we notice that $r+I = (r(1-e)+I)+(re+I)$ for each $r \in R$ Also $\overline{x} = r(1-e)+I = se+I$ means that $r - re = (s+r')e$ for some $r' \in R$ which shows that $\overline{x} = \overline{0}$. Now R/I, being the direct sum of two pure Baer injective R-modules, should be pure Baer injective. □

DEFINITION 2.4 [8]. A left R-module M is called pure-split if every pure submodule of M is a direct summand. R is left pure-split if ${}_RR$ is pure-split.

It obviously follows that every left pure-split ring is left pure hereditary.

It is easily seen that every pure submodule of a pure-split module is pure-split, and the quotient of a pure-split module by a pure submodule is again pure-split.

We thus extract the following simple result

THEOREM 2.5. For a given ring R the following statements are equivalent.

(1) Every direct sum of copies of R is pure-split;

(2) Every flat R-module is pure-split.

PROOF. (1) ⇒ (2) For any flat module M, there is a pure exact sequence

$$0 \to K \to F \to M \to 0$$

where F is free and so is a pure-split module; consequently both K and M are pure-split modules.

(2) ⇒ (1) Clear. □

THEOREM 2.6. The following statements are pair-wise equivalent for a given ring R·

(1) R is left pure-split;

(2) Every left R-module is pure Baer injective;

(3) Every pure left ideal of R is pure Baer injective;

(4) Every pure left ideal of R is principal.

PROOF. (1) ⇒ (2) Let I be a pure left ideal of R and let $f : I \to M$ be an R-homomorphism. Then $R = I \oplus J$ for some left ideal J of R, so that there is an R-homomorphism $R \to M$ that extends f

(2) ⇒ (3) Obvious.

(3) ⇒ (1) Since every pure left ideal of R is pure Baer injective, each pure exact sequence

$$0 \to I \to R \to R/I \to 0$$

splits, showing that R is left pure-split.

(4) $\Rightarrow$ (1) Let M be an R-module and let $I = Rx$ be a pure left ideal of R. So, for some $a \in I$, $x = xa$. If $f : Rx \to M$ is a given R-homomorphism, the restriction on I of the R-homomorphism $\overline{f} : R \to M$ effected by $\overline{f}(1) = f(a)$ is f. Indeed $\overline{f}(rx) = f(rxa) = f(rx)$ for all $r \in R$.

(1) $\Rightarrow$ (4) Obvious. □

REMARK 2.7. We know that ${}_RR$ is semisimple if and only if every R-module is semisimple. This fact cannot be extended to pure-splitting. To see this we notice that $\mathbb{Z}$ is a pure-split $\mathbb{Z}$-module However $\mathbb{Z}$ can be embedded as a pure submodule in a pure injective $\mathbb{Z}$-module J. But J cannot be pure-split since otherwise $\mathbb{Z}$ will be a direct summand of J, contradicting the fact that $\mathbb{Z}$ is not pure injective. □

However, the previous property is valid for certain types of rings. For example, a regular ring R is pure-split if and only if it is semisimple. We can thus state the following proposition.

PROPOSITION 2.8. For a regular ring R, the following statements are equivalent:

(1) R is pure -split;

(2) Every module is semisimple;

(3) Every module is pure-split;

(4) Every (pure) exact sequence is split exact.

PROOF. Clear. □

We now recall the following definition; for analogous and related concept cf. [6, p. 48].

DEFINITION 2.9. An R-module M is called pure injective relative to the R-module N, or simply N-pure injective, if in each diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \xrightarrow{f} & N \\ & & g \downarrow & & \\ & & M & & \end{array}$$

where the embedding $f(K)$ is pure in N, there exists an R-homomorphism $h : N \to M$ such that $g = hf$.

The following theorem relates some of the previous notions.

THEOREM 2.10. For a left perfect ring R the following properties hold:

(1) Every flat R-module is pure-split,

(2) Every R-module is pure Baer injective, and

(3) Every R-module is pure injective relative to any flat R-module.

PROOF. (1) and (2) are both direct. To prove (3), let C be a left R-module and N be a pure submodule of a flat R-module M. Then N is a direct summand of M, so that every homomorphism $N \to C$ extends to a homomorphism $M \to C$. □

DEFINITION 2.11 (see [9]). A non-zero R-module M is called pure simple if $\{0\}$ and M are its only pure submodules.

PROPOSITION 2.12. In a commutative ring R in which every ideal is the intersection of maximal pure ideals, every epimorphism $f : I \to S$ whose domain is a pure ideal of R and codomain is a pure simple R-module has an extension $\overline{f} : R \to S$.

PROOF. Let $f : I \to S$ be as given in the premise By assumption Ker f is the intersection of a family $\{C_j : j \in J\}$ of maximal pure ideals of R. If $I \subseteq C_j$ for all $j \in J$, then Ker $f = I$ and this means that $S = \{0\}$ contradicting that S is a pure simple R-module. Thus for some $j \in J$ we have $I \not\subseteq C_j$ Now R is commutative. So, $I + C_j$ is pure; and maximality of C_j forces $R = I + C_j$. But $I \cap C_j$ is

pure in R. So $I \cap C_j$ is pure in I, and so $I \cap C_j/\text{Ker}\, f$ is a pure submodule of $I/\text{Ker}\, f \cong S$ By assumption either $I \cap C_j/\text{Ker}\, f = \{0\}$, or $I \cap C_j/\text{Ker}\, f = I/\text{Ker}\, f$. The latter assumption is impossible, since otherwise $I \subseteq C_j$. So, $I \cap C_j = \text{Ker}\, f$. Now the assignment $\overline{f} : R \to S$ defined by $\overline{f}(r) = f(i)$ where $i \in I$ that satisfies $r = i + c$ for some $c \in C_j$ is a well-defined function that gives the required homomorphism. □

COROLLARY 2.13. Let R be a commutative ring in which every ideal is the intersection of maximal pure ideals. Then

(1) Every pure simple R-module is pure Baer injective,

(2) Every semisimple R-module is a direct sum of pure Baer injective R-modules.

PROOF. The proof follows from Proposition 2 12. □

LEMMA 2.14. A finitely generated non-zero R-module M possesses a maximal pure submodule

PROOF. The set $\Lambda = \{K : K < M \text{ and } K \text{ pure in } M\}$ is partially ordered by inclusion. The union of a chain of Λ is clearly a member of Λ; and an appeal to Zorn's Lemma yields the result. □

THEOREM 2.15. Let M be an R-module M in which every cyclic submodule is pure-split, then every non-zero submodule of M contains a pure simple submodule.

PROOF. Let $\{0\} \neq K \leq M$, if $0 \neq x \in K$. Then by Lema 2.14 Rx contains a maximal pure submodule, say, H. Thus $Rx = H \oplus H'$. This shows that H' is a non-zero pure simple submodule in Rx and so is a non-zero pure simple submodule in K. □

It seems that an appropriate notion of an "intersection property" would play an important role in the structure theory of rings. Thus within our context, we define

DEFINITION 2.16. An R-module M is said to have the pure intersection (resp. pure finite intersection) property if and only if the intersection of any (resp. finite) family of pure submodules of M is again pure.

It can be easily shown that any commutative ring possesses the pure finite intersection property, and furthermore, a regular ring R possesses the pure intersection property to each R-module

PROPOSITION 2.17. Any torsion-free R-module M over a Prüfer ring R has the pure intersection property.

PROOF. It is known that over a Prüfer ring purity and RD-purity are equivalent notions, see [4, p. 706]. Thus the required result is immediate if we notice that the intersection of any family of RD-submodules of a torsion-free module is again an RD-submodule, see [10, p. 39]. □

PROPOSITION 2.18. A torsion-free R-module M over a principal right ideal ring R has the pure intersection property.

PROOF. Let $\{N_j : j \in J\}$ be a family of pure submodules of M. In view of the torsion-freeness of M, $K = \cap_j N_j$ is an RD-submodule, see [10, p. 39]. We prove first that K is an $\cap$-pure submodule. To this end let I be a given right ideal of R. Now K is RD-pure and R is a principal right ideal ring So, given $x \in K \cap IM$, we can find $a \in I$ and $m \in M$ such that $x = am \in aK$. Thus $K \cap IM = IK$, showing that K is $\cap$-pure. Finally the flatness of M guarantees the purity of K. □

3. THE $SSBI$-RING AND Σ-PURE BAER INJECTIVE MODULE

In this section we introduce two related notions namely $SSBI$-ring and Σ-pure injective module which prove to be useful to our investigations.

Byrd [11] calls a ring R an SSI-ring if every semisimple R-module is injective. In what follows we generalize this concept.

DEFINITION 3.1. A ring R is called an $SSBI$-ring if every semisimple R-module is pure Baer injective.

THEOREM 3.2. A ring which is both a V-ring and $SSBI$-ring satisfies the ascending chain condition (A.C.C.) on pure left ideals.

PROOF. Let $I_0 \subset I_1 \subset \ldots \subset I_m \subset \ldots$ be a strictly ascending chain of pure left ideals of R and for a given I_k in the chain take $a \in I_k$ such that $a \notin I_{k-1}$. The family $\Lambda_k = \{L \leq R : I_{k-1} \leq L < I_k : a \notin L\}$ is partially ordered by inclusion in which every chain has an upper bound. Let L_k be a maximal member of Λ_k. Thus, $Ra + L_k/L_k$ is simple and the given assumption yields $I_k/L_k = Ra + L_k/L_k \oplus N/L_k$ for some left ideal N of R. Now, since $a \notin N$, the maximality of L_k forces $N = L_k$. This means that I_k/L_k is simple. Hence $I/L_k = I_k/L_k \oplus N_k/L_k$ for some left ideal N_k and $I = \bigcup_K I_k$. Now, if $\overline{x} = x + L_k = \overline{x}_k + \overline{n}_k$ for some $\overline{x}_k \in I_k/L_k$ and $\overline{n}_k \in N_k/L_k$, the assignment $\sigma : I \to \oplus_k I_k/L_k$, with $\sigma(x) = \{\overline{x}_k\}$ is a well-defined R-homomorphism. For if $x \in I$, then $x \in L_i$ for some i and $x \in L_{i+j}$ for all j. So, $\sigma(x) = \{\overline{x}_k\} \in \oplus_k(I_k/L_k)$ since $\overline{x}_{i+j} = 0$ and $\oplus_k(I_k/L_k)$ is semisimple. By hypothesis $\oplus_k(I_k/L_k)$ is therefore pure Baer injective and we should have the extension R-homomorphism $\overline{\sigma} : R \to \oplus_k(I_k/L_k)$ of σ. Furthermore, since $\overline{\sigma}(1) \in \oplus_{i=1}^n(I_i/L_i)$ for some $n \in \mathbb{Z}^+$, we see that $\overline{\sigma}(I) \subseteq \oplus_{i=1}^n(I_i/L_i)$. Suppose now that $\overline{x} = x + I_{n+1} \in I_{n+1}/L_{n+1}$ and $\sigma(x) = (\overline{x}_k)$, then $\overline{x}_{n+1} = \overline{x} = 0$. The argument shows that $I_{n+1} = L_{n+1}$, contradicting the fact $L_{n+1} \in \Lambda_{n+1}$. This shows that the above tower of pure left ideals is of finite length. □

DEFINITION 3.3. A left R-module M is called Σ-pure Baer injective if every direct sum of copies of M is pure Baer injective.

As examples of Σ-pure Baer injective modules we mention torsion-free modules, modules over integral domains or over left pure-split rings.

THEOREM 3.4. A ring R in which every injective module is Σ-pure Baer injective, satisfies the ascending chain condition on pure left ideals.

PROOF. Let $I_1 \subset I_2 \subset \ldots \subset I_n \subset \ldots$ be a chain of pure left ideals For each i let K_i be the injective hull of R/I_i and let $K = \oplus_i K_i$. For every $\sigma \in \mathbb{Z}^+$, $\Pi_i K_i = K_\sigma \oplus \Pi_{i \neq \sigma} K_i$. If we set $M_\alpha = \Pi_i K_i$, then M_α is injective By abuse of notation we have

$$\oplus_{\alpha \in \mathbb{Z}^+} M_\alpha = (\oplus_{\alpha \in \mathbb{Z}^+} K_\alpha) \oplus \left(\oplus_{\alpha \in \mathbb{Z}^+} \prod_{i \neq \alpha} K_i \right).$$

By assumption $\oplus_\alpha M_\alpha$ is pure Baer injective. Thus K itself is pure Baer injective. Now the R-homomorphism $f : \cup_i I_i \to K$, defined by $f(x) = \{x + I_i\}$ extends to an R-homomorphism $\overline{f} : R \to K$. Let $n \in \mathbb{Z}^+$ such that $\overline{f}(1) \in \oplus_{i=1}^n K_i$. Then $f(\cup_i I_i) \leq \oplus_{i=1}^n K_i$. So, if $x \in \cup_i I_i$ then $x \in I_\alpha$ for all $\alpha > n$, and so $\cup_i I_i = I_{n+1}$ and the chain should terminate. □

PROPOSITION 3.5. A direct summand of a Σ-pure Baer injective module is again Σ-pure Baer injective.

PROOF. Immediate. □

THEOREM 3.6. R is left pure-split if and only if R is left pure hereditary and Σ-pure Baer injective.

PROOF. $\Leftarrow$ Let I be a pure left ideal of R. Then I is projective and so a direct summand of a free R-module F. But R is pure Baer injective. Thus both F and I are pure Baer injective, yielding the left pure-splitting of R. $\Rightarrow$ The proof follows from Theorem 2.6. □

4. P-ESSENTIAL SUBMODULES

In this section we introduce the notion of p-essential submodules with the aim of gaining further insight about the structure of pure Baer injective modules. For example, Corollary 4.16 to follow presents a criterion for a left R-module over a commutative ring R to be pure Baer injective, employing this notion of p-essentiality. Also by using this notion, corollary 4 17 sharpens a result [5] of Kaplansky's on V-rings.

DEFINITION 4.1. A submodule K of an R-module M is called p-essential in M, abbreviated by $K \trianglelefteq' M$, in case for every pure submodule L of M, $K \cap L = \{0\}$ implies that $L = \{0\}$. In this case

M is called a p-essential extension of K. Also a monomorphism $f : N \to M$ is called a p-essential if $Im\, f \trianglelefteq' M$.

Given left R-modules M and $N, PH_R(M,N)$ designates the set of R-homomorphisms $PH_R(M,N) = \{h : h : M \to N$ and $\operatorname{Ker} h$ pure in $M\}$.

PROPOSITION 4.2. The following statements are equivalent for any submodule K of an R-module M

(1) $K \trianglelefteq' M$;

(2) For each R-module N and for each $h \in PH_R(M,N)$

$$(\operatorname{Ker} h) \cap K = \{0\} \text{ implies that } h \text{ is monomorphism}$$

PROOF. (1) $\Rightarrow$ (2) Holds by the definition of p-essential submodule.

(2) $\Rightarrow$ (1) Since any pure submodule L of M is the kernel for some $h \in PH_R(M,A)$ for some R-module A, then $L \cap K = \{0\}$ implies that $L = \{0\}$ and so $K \trianglelefteq' M$. □

COROLLARY 4.3. A monomorphism $f : K \to M$ is p-essential if and only if (any epimorphism) $h \in PH_R(M, -)$ is monic whenever hf is monic

PROOF. Let $0 \to K \xrightarrow{f} M$ be p-essential and $h : M \to N$, where $h \in PH_R(M,N)$. Then if hf is monic $\operatorname{Ker} h \cap Im\, f = \{0\}$. So, $\operatorname{Ker} h = \{0\}$ and h is monic. Conversely, suppose that $f : K \to M$ is a monomorphism satisfying the given condition and let L be a pure submodule of M with $L \cap Im\ f = \{0\}$. Then πf is obviously monic, π being the canonical map $\pi : M \to M/L$. By assumption π should be monic. This means that $L = \{0\}$ and $Im\, f \trianglelefteq' M$. □

The following result is analogous to a similar result concerning essential submodules of a module.

THEOREM 4.4. Let $K \leq N \leq M$ be a tower of R-modules. Then:

(1) If $K \trianglelefteq' M$ then $N \trianglelefteq' M$.

(2) If N is pure in M and $K \trianglelefteq' M$, then $K \trianglelefteq' N$ and $N \trianglelefteq' M$.

(3) If M has pure finite intersection property and if N is pure in M, then $K \trianglelefteq' M$ if and only if $K \trianglelefteq' N$ and $N \trianglelefteq' M$.

PROOF. (1) and (2) are obvious.

(3) Let L be pure in M with $L \cap K = \{0\}$. By assumption $L \cap N$ is pure in M. This means that $L \cap N$ is pure in N. Thus $L \cap N = \{0\}$ and consequently $L = \{0\}$. Therefore $K \trianglelefteq' M$ □

COROLLARY 4.5. Let M be an R-module that has the pure finite intersection property. If H is pure in M, then $H \cap K \trianglelefteq' M$ if and only if $H \trianglelefteq' M$ and $K \trianglelefteq' M$ for any submodule K of M.

PROOF. $\Rightarrow$ The proof follows from Theorem 4.4. $\Leftarrow$ Suppose that $H \trianglelefteq' M$ and $K \trianglelefteq' M$. Given a pure submodule L of M with $L \cap (H \cap K) = \{0\}$, then $L \cap H$ is pure in M by hypothesis. Thus $L \cap H = \{0\}$ and consequently $L = \{0\}$. □

EXAMPLE 4.6. We give here an example of submodules A, B, A' and B' of a certain $\mathbb{Z}$-module $M = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ with $A \trianglelefteq' B$ and $A' \trianglelefteq' B'$ whereas $A + A'$ is not p-essential in $B + B'$. The idea of this counter example is lifted from example 1.2 from Goodearl's monograph [7]. Take $A = A' = \mathbb{Z}(2,\bar{0})$, the submodule generated by $(2,\bar{0}), B = \mathbb{Z}(1,\bar{0})$ and $B' = \mathbb{Z}(1,\bar{1})$. Now $A \cap \mathbb{Z}(0,\bar{1}) = \{0\}$; see Goodearl [7]. What is left now to prove our assertion is to prove that $\mathbb{Z}(0,\bar{1})$ is pure in $B + B'$ To see this suppose that $n(m,\bar{0}) + (k,\bar{k}) = (0,\bar{1})$. This means that L is odd and $m = -k$. Again n is odd This gives $n(m,\bar{0}) + (k,\bar{1}) = n(0,\bar{1})$, showing that $\mathbb{Z}(0,\bar{1})$ is pure in $B + B'$. □

A pure left ideal I of a ring R is a direct summand of R if I is pure Baer injective. Hence, we have the following:

PROPOSITION 4.7. A ring R cannot have a proper pure left ideal I which is both p-essential and pure Baer injective.

PROOF. Clear. □

PROPOSITION 4.8. A pure injective module N does not have a proper p-essential extension M in which N is pure.

PROOF. Clear. □

COROLLARY 4.9. Any R-module M cannot have a proper submodule which is both injective and p-essential.

PROOF. Any injective submodule K of M is pure in M. But K is pure injective. So, if K is p-essential in M, the previous proposition shows that $K = M$. □

DEFINITION 4.10. A submodule N of an R-module M is called a pure closed submodule of M if M does not contain a proper p-essential extension of N. Obviously $N \leq M$ is pure closed if and only if $N \trianglelefteq' K \leq M$ implies that $K = N$.

PROPOSITION 4.11. Any direct summand of an R-module M is pure closed.

PROOF. Let $M = A \oplus B$. If $A \trianglelefteq' K \leq M$, then $K \cap B$ is pure in K But $K \cap B \cap A = \{0\}$ This means that $K \cap B = \{0\}$; and so $K = A$. □

COROLLARY 4.12. (1) Every pure injective R-module M is pure closed in any R-module that contains M as a pure submodule.

(2) A pure left ideal of R which is pure Baer injective is pure closed in R.

PROOF. (1) Let M be embedded in N as a pure submodule. In this case M is a direct summand of N; and so M should be pure closed in M

(2) Clear. □

DEFINITION 4.13. Let N and K be submodules of an R-module M with K pure in M K is called pure relative complements of N in M if K is maximal with the property $K \cap N = \{0\}$.

PROPOSITION 4.14. Every R-submodule of M has a pure relative complement in M.

PROOF. Let N be a given submodule of M and consider the set

$$\Lambda = \{K; K \leq M, K \text{ pure in } M \text{ and } N \cap K = \{0\}\}.$$

Λ is partially ordered by inclusion. Obviously any chain of Λ has an upper bound. Zorn's lemma then guarantees that Λ has a maximal member, which means that N has a pure relative complement in M □

PROPOSITION 4.15. I and J are given ideals of a commutative ring R If J is pure relative complements of I in R, then $I \oplus J$ is p-essential in R.

PROOF. Let $A \cap (I \oplus J) = \{0\}$ for some pure ideal A of R Then $I \cap (A \oplus J) = \{0\}$ Since R is commutative, $A \oplus J$ is pure in R. The maximality of J forces $A = \{0\}$. This means that $I \oplus J$ is p-essential in R. □

So we deduce that a commutative ring that has no proper p-essential ideals is necessarily semisimple.

COROLLARY 4.16. R is a commutative ring. An R-module M is pure Baer injective if and only if $\mathrm{Hom}_R(R, M) \to \mathrm{Hom}_R(J, M)$ is an epimorphism for every pure, p-essential ideal J of R.

PROOF. Consider a homomorphism $f : I \to M$ where I is a pure ideal of R By Proposition 4.14 we can find a pure relative complement J of I in R. Then f extends to a homomorphism $I \oplus J \to M$, and this extends to a homomorphism $R \to M$ by our assumption.

It is known [5] that if R is a right V-ring, then $I^2 = I$ for any right ideal I of R. This yields the celebrated result of Kaplansky stating that a commutative ring is a V-ring if and only if it is regular The following result refines that of Kaplansky's.

COROLLARY 4.17. A commutative ring R is a V-ring if and only if $I = I^2$ for every p-essential ideal I of R.

PROOF. Let J be an ideal of R. If J' is a pure relative complement of J in R, then $J + J'$ is p-essential. By assumption $(J + J')^2 = J + J'$. But since $J \cap J' = \{0\}$, we get $J + J' = J^2 + J'^2$ This directly gives $J = J^2$, showing that R is a V-ring. □

ACKNOWLEDGMENT. This work comprises a portion of a Ph.D. thesis written under the supervision of Professor Ismail A. Amin, Professor Javed Ahsan and Professor Mohammed Seoud. The author wishes to thank them all for suggesting this problem, and also for some helpful comments throughout the development of this work.

REFERENCES

[1] FAITH, C., *Algebra II: Ring Theory*, Springer-Verlag, Berlin-Heidelberg-New York, 1976.

[2] FIELDHOUSE, D.J., Pure theories, *Math. Ann.*, **184** (1969), 1-18, MR 40#569.

[3] FIELDHOUSE, D.J., Regular rings and modules, *J. Austral. Math. Soc.*, **13** (1972, 477-491

[4] WARFIELD, R.B., Jr., Purity and algebraic compactness for modules, *Pacific J. Math.*, **28** (1969), 699-719.

[5] ANDERSON, F.W. and FULLER, K.R., *Rings and Categories of Modules*, Springer-Verlag, Berlin-Heidelberg-New York, 1973.

[6] STENSTRÖM, B., *Rings of Quotients, Springer-Verlag*, Berlin-Heidelberg-New York, 1975.

[7] GOODEARL, K.R., *Ring Theory, Nonsingular Rings and Modules*, Marcel Dekker, Inc., New York, 1976.

[8] AZUMAYA, G. and FACCHINI, A., Rings of pure global dimension zero and Mittag-Leffler modules, *J. Pure Appl. Algebra*, **62** (1989), 109-122.

[9] FIELDHOUSE, D.J., Pure simple and indecomposable rings, *Canad. Math. Bull.*, **13** (1969), 71-78.

[10] FUCHS, L. and SALCE, L., *Modules Over Valuation Domains, Lecture Notes in Pure and Applied Mathematics*, Vol. 97, Marcel Dekker, Inc., New York, 1985

[11] BYRD, K.A., Rings whose quasi-injective modules are injectives, *Proc. Amer. Math. Soc.*, **33** (1972), 235-240.