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FOURIER TRANSFORM OF $h_n(x+p)h_n(x-p)$

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ABSTRACT. We evaluate Fourier transform of a function with Hermite polynomials involved. An elementary proof is based on a combinatorial formula for Hermite polynomials.

KEY WORDS AND PHRASES: Fourier transform, Hermite polynomials, Hermite functions.

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In this note, by using elementary means, we prove the identity

$$\int_{-\infty}^{\infty} e^{-i\sqrt{2}qx} h_n(x+\frac{p}{\sqrt{2}})h_n(x-\frac{p}{\sqrt{2}})dx = e^{-(p^2+q^2)/2}L_n(p^2+q^2) \tag{1}$$

which was previously known in special cases $p=0$ or $q=0$, cf. [3, p.503 (10)] and [1, Section 1.10, (10)]. In the above formula $h_n(x)$, $n=0,1,\dots$, are the normalized Hermite functions

$$h_n(x) = \left(2^n\pi^{1/2}n!\right)^{-1/2}e^{-x^2/2}H_n(x),$$

$H_n(x)$ denotes the n-th Hermite polynomial, [5, p. 102] and $L_n^\alpha(x)$ denotes the n-th Laguerre polynomial of order α, [5, p. 97]. When $\alpha=0$ we write $L_n(x)$ rather than $L_n^0(x)$. The proof of (1) reduces to showing the identity

$$\int_{-\infty}^{\infty} e^{-i\sqrt{2}qx} H_n(x+\frac{p}{\sqrt{2}})H_n(x-\frac{p}{\sqrt{2}})e^{-x^2}dx = 2^n\pi^{1/2}n!\,e^{-q^2/2}L_n(p^2+q^2). \tag{2}$$

The proof of (2) is based, in turn, on the combinatorial formula

$$H_n(x+\frac{p}{\sqrt{2}})H_n(x-\frac{p}{\sqrt{2}}) = 2^n n!\sum_{j=0}^{n}\frac{1}{2^j j!}L_{n-j}^{-1}(p^2)H_j(x)^2 \tag{3}$$

which will be proved in a moment (we were not able to find the above formula in the literature). Substituting (3) into the integral to be evaluated in (2) and using the known values of the integral in (2) for $p = 0$ and $j = 0, 1, \dots$, we get

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{-i\sqrt{2}qx} H_n(x + \frac{p}{\sqrt{2}}) H_n(x - \frac{p}{\sqrt{2}}) e^{-x^2} dx \\
&= 2^n n! \sum_{j=0}^{n} \frac{1}{2^j j!} L_{n-j}^{-1}(p^2) \int_{-\infty}^{\infty} e^{-i\sqrt{2}qx} H_j(x)^2 e^{-x^2} dx \\
&= 2^n \pi^{1/2} n! e^{-q^2/2} \sum_{j=0}^{n} L_{n-j}^{-1}(p^2) L_j(q^2) \\
&= 2^n \pi^{1/2} n! e^{-q^2/2} L_n(p^2 + q^2).
\end{aligned}$$

The last step required using the well-known identity

$$\sum_{j=0}^{n} L_{n-j}^{a}(x) L_j^{b}(y) = L_n^{a+b+1}(x + y)$$

with $a = -1$ and $b = 0$.

We were motivated to consider the integral (1) by the interesting work of Strichartz [4] where he considered the Hermite functions

$$\varphi_{\alpha,\beta,\epsilon}(\overline{q}, \overline{p}) = \pi^{-m/2} \int_{\mathbf{R}^m} e^{i\epsilon\sqrt{2}\overline{q}\overline{x}} h_\beta(\overline{x} + \frac{\overline{p}}{\sqrt{2}}) h_\alpha(\overline{x} - \frac{\overline{p}}{\sqrt{2}}) d\overline{x}.$$

In the above formula $\overline{p}, \overline{q} \in \mathbf{R}^m$, $\alpha, \beta \in \mathbf{Z}_+^m$, $\epsilon = \pm 1$ and

$$h_\alpha(\overline{x}) = \Pi_{j=1}^{m} h_{\alpha_j}(x_j)$$

for $\overline{x} = (x_1, \dots, x_m)$ and $\alpha = (\alpha_1, \dots, \alpha_m)$. It is clear that (1) gives

$$\varphi_{\alpha,\alpha,\epsilon}(\overline{q}, \overline{p}) = \pi^{-m/2} \exp\big(- \frac{|\overline{p}|^2 + |\overline{q}|^2}{2} \big) \Pi_{j=1}^{m} L_{\alpha_j}(p_j^2 + q_j^2).$$

Let us also add that a group-theoretic approach allows to find an explicit form of $\varphi_{\alpha,\beta,\epsilon}(\overline{q}, \overline{p})$, with arbitrary α, β, cf. [2, p.64].

The rest of the note is devoted to the proof of (3) which we now write in the form

$$H_n(x + y) H_n(x - y) = 2^n n! \sum_{j=0}^{n} \frac{1}{2^j j!} L_{n-j}^{-1}(2y^2) H_j(x)^2. \tag{4}$$

Applying Mehler's formula for Hermite polynomials

$$\sum_{n=0}^{\infty} \frac{1}{2^n n!} H_n(x) H_n(y) t^n = (1 - t^2)^{-1/2} \exp\left(\frac{2xyt - x^2t^2 - y^2t^2}{1 - t^2} \right), \qquad |t| < 1,$$

and the generating function formula for Laguerre polynomials

$$\sum_{k=0}^{\infty} L_k^{\alpha}(x) t^k = (1 - t)^{-\alpha-1} \exp\left(\frac{-xt}{1 - t} \right), \qquad |t| < 1,$$

for $\alpha = -1$ one gets

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{2^n n!} H_n(x + y) H_n(x - y) t^n &= \exp\left(- 2\frac{y^2 t}{1 - t} \right) \cdot (1 - t^2)^{-1/2} \exp\left(\frac{2t}{1 + t} x^2 \right) \\
&= \left(\sum_{k=0}^{\infty} L_k^{-1}(2y^2) t^k \right) \cdot \left(\sum_{m=0}^{\infty} \frac{H_m(x)^2}{2^m m!} t^m \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} 2^j j! L_{n-j}^{-1}(2y^2) H_j(x)^2 \right) t^n.
\end{aligned}$$

Comparing the coefficients gives (3). Note that an application of the Cauchy multiplication of the two series above is being possible by the fact that both are, for fixed y and x, absolutely convergent for $|t| < 1$. This is easily seen once we use the well known estimate $L_k^{-1}(2y^2) = O(k^{-3/4})$, cf. [5, (7.6.10)], and a similar estimate for $H_m(x)$.

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