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ON α-CONVEX FUNCTIONS OF ORDER β

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ABSTRACT. In 1969 Mocanu [1] introduced and studied a new class of analytic functions consisting of α-convex functions Many mathematicians have studied and shown the properties of this class Now we will define new classes like that Mocanu class and then investigate and give some results The class of α-convex functions of order β partially includes Mocanu's class

KEY WORDS AND PHRASES: α-convex function, starlike function, convex function
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1. INTRODUCTION

Let U be the unit disc in the complex plane, $U = \{z \in C : |z| < 1\}$ and let Λ be a class of analytic functions, $f(z) = z + a_2z^2 + \ldots$ in U We denote three subclasses of Λ, as S, S^* and K, the sets of univalent functions, starlike functions and convex functions, respectively Also we denote by $S^*(\alpha)$ and $K(\alpha)$ the classes of starlike functions of order α and convex functions of order α which are analytically expressed as follows.

$$S^*(\alpha) = \left\{f(z) \in A\,;\, \mathrm{Re}\,\frac{zf'(z)}{f(z)} > \alpha,\ \ z \in U\right\}, \tag{1 1}$$

$$K(\alpha) = \left\{f(z) \in A\,;\, \mathrm{Re}\left(1 + \frac{zf'(z)}{f'(z)}\right) > \alpha,\ \ z \in U\right\}, \tag{1 2}$$

where α is a real number and $1 > \alpha \geq 0$ We can put $S^*(0) = S^*$ and $K(0) = K$ It is well-known that $K \subset S^* \subset S \subset A$, $S^*(\alpha_1) \subset S^*(\alpha_2)$, and $K(\alpha_1) \subset K(\alpha_2)$ for $0 \leq \alpha_2 \leq \alpha_1 < 1$.

The following result was shown by Marx [2] independently of Strohhäcker [3]

THEOREM A. It holds that $K \subset S^*\left(\frac{1}{2}\right)$.

In 1969 Mocanu [1] defined a new subclass of A consisting of α-convex functions as follows

DEFINITION 1. A function $f(z) \in A$ is said to be an α-convex function if $f(z)$ satisfies that $\frac{f(z)}{z} \cdot f'(z) \neq 0$ and

$$\mathrm{Re}\left\{\alpha\left(1 + \frac{zf'(z)}{f'(z)}\right) + (1-\alpha)\frac{zf'(z)}{f(z)}\right\} \geq 0 \quad z \in U, \tag{1 3}$$

for some real number α

We denote by $M(\alpha)$ the class of α-convex functions and we often call it Mocanu's class Miller, Mocanu and Reade [4] showed the following Theorem B

THEOREM B. It holds that $M(\alpha) \subset S^*$ for any real number α and $M(\alpha) \subset K \subset S^*$ for $\alpha \geq 1$

We will give two remarks about Definition 1

REMARK 1. If the condition (1 3) is satisfied, then the condition $\frac{f(z)}{z} \cdot f'(z) \neq 0$ is always true, so this latter condition is not needed This fact is founded in [4]

REMARK 2. The equality in (1 3) does not appear, because we have to consider the minimal principle of harmonic function in the open unit disc U

We will define new subclasses of A

DEFINITION 2. A function $f(z) \in A$ is said to be an α-convex function of order β of type 1 if $f(z)$ satisfies

$$\operatorname{Re}\left\{\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\alpha)\frac{zf'(z)}{f(z)}\right\} > \beta \quad z \in U, \tag{1 4}$$

for some real α and β

DEFINITION 3. A function $f(z) \in A$ is said to be an α-convex function of order β of type 2 if $f(z)$ satisfies

$$\operatorname{Re}\left\{\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\alpha)\frac{zf'(z)}{f(z)}\right\} < \beta \quad z \in U, \tag{1 5}$$

'for some real α and β

We denote by $M(\alpha, \beta)$ and $N(\alpha, \beta)$ the classes of α-convex functions of order β of type 1 and of type 2, respectively We can put $M(\alpha, 0) = M(\alpha)$. We note that it must be $1 > \beta$ in Definition 2 and $1 < \beta$ in Definition 3

2. LEMMA AND THEOREMS

To prove our results we need the following lemma which is a special case of Corollary 1 in [5]

LEMMA Let $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ be analytic in U If there exists a point $z_0 \in U$ such that $\operatorname{Re} p(z) > \gamma$ for $|z_0| < |z|$ and $\operatorname{Re} p(z_0) = \gamma$, then we have

$$\operatorname{Re}\frac{z_0 p'(z_0)}{p(z_0)} \begin{cases} \leq \dfrac{\gamma-1}{2\gamma} \left(1 > \gamma \geq \dfrac{1}{2}\right) & (2\ 1) \\ \leq \dfrac{2\gamma}{\gamma-1} \left(\dfrac{1}{2} \geq \gamma \geq 0\right) & (2\ 2) \end{cases}$$

Now we will show two theorems After that using the above lemma, we will give only the proof of Theorem 1, because we can easily give the proof of Theorem 2 in almost the same was as in Theorem 1

THEOREM 1. It holds that $M(\alpha, \beta) \subset S^*(\gamma)$ if it is satisfied that

$$\gamma + \alpha\frac{\gamma-1}{2\gamma} \leq \beta, \quad \left(1 > \gamma \geq \frac{1}{2}\right) \tag{2 3}$$

$$\gamma + \alpha\frac{\gamma}{2(\gamma-1)} \leq \beta \quad \left(\frac{1}{2} \geq \gamma \geq 0\right) \tag{2 4}$$

for $\alpha > 0$ and $1 > \beta$

THEOREM 2. It holds that $N(\alpha, \beta) \subset S^*(\gamma)$ if it is satisfied that

$$\gamma + \alpha\frac{\gamma-1}{2\gamma} \geq \beta, \quad \left(1 > \gamma \geq \frac{1}{2}\right) \tag{2 5}$$

$$\gamma + \alpha\frac{\gamma}{2(\gamma-1)} \geq \beta \quad \left(\frac{1}{2} \geq \gamma \geq 0\right) \tag{2 6}$$

for $\alpha < 0$ and $1 < \beta$

PROOF OF THEOREM 1. Let us put $P(z) = \frac{zf'(z)}{f(z)}$ and

$$W(z) = \alpha \cdot \left(1 + \frac{zf'(z)}{f'(z)}\right) + (1-\alpha) \cdot \frac{zf'(z)}{f(z)} = p(z) + \alpha \frac{zp'(z)}{p(z)} .$$

Then $f(z) \in M(\alpha, \beta)$ implies

$$\operatorname{Re} W(z) = \operatorname{Re}\left\{p(z) + \frac{zp'(z)}{p(z)}\right\} > \beta, \quad z \in U. \tag{2 7}$$

Now we must show $\operatorname{Re} p(z) > \gamma$ for $z \in U$ It is true in the neighborhood of $z = 0$ So if there exists a point $z_0 \in U$ such that $\operatorname{Re} p(z) > \gamma$ for $|z| < |z_0|$ and $\operatorname{Re} p(z_0) = \gamma$, then, by the result of our lemma and $\alpha > 0$, we have

$$\operatorname{Re} W(z_0) = \operatorname{Re} p(z_0) + \operatorname{Re} \alpha \frac{z_0 p'(z_0)}{p(z_0)} \leq \gamma + \alpha \frac{\gamma - 1}{2\gamma} \leq \beta,$$

for $1 > \gamma \geq \frac{1}{2}$ This contradicts the assumption (2 7)

In the same way we have $\operatorname{Re} W(z_0) \leq \gamma + \alpha \frac{\gamma}{2(\gamma-1)} \leq \beta$, for $\frac{1}{2} \geq \gamma \geq 0$ which leads to the contradiction Thus there exists no such a point z_0 in U This completes the proof of Theorem 1 □

3. APPLICATIONS

We will show some applications of Theorem 1 and Theorem 2

First of all, we put $\beta = 0$ in (2 3) and (2 4) of Theorem 1 Then we have

COROLLARY 1. It holds that

$$M(\alpha) \subset S^*(\gamma) \quad \text{for} \quad \frac{1}{2} \leq \gamma \leq \frac{-\alpha + \sqrt{(\alpha^2 + 8\alpha)}}{4} \, (\alpha \geq 1) \tag{3 1}$$

$$M(\alpha) \subset S^*(\gamma) \quad \text{for} \quad 1 - \frac{\alpha}{2} \leq \gamma \leq \frac{1}{2} \quad (2 \geq \alpha \geq 1). \tag{3 2}$$

COROLLARY 2. Theorem A is true, that is, $K \subset S^*\left(\frac{1}{2}\right)$

Putting $\alpha = 1$ in Corollary 1 above we have $\gamma = \frac{1}{2}$, so we can easily see that $K \subset S^*\left(\frac{1}{2}\right)$ from $M(1) = K$

COROLLARY 3. It holds that $M(\alpha, \beta) \subset S^*$ for any real numbers α and $1 > \beta \geq 0$

When we put $\gamma = 0$ in (2 2) of the lemma, we have Re $\frac{z_0 p'(z_0)}{p(z_0)} \leq 0$ So we have Re $W(z_0) =$ Re $p(z_0) + \alpha \operatorname{Re} \frac{z_0 p'(z_0)}{p(z_0)} \leq 0$ Since the assumption of Theorem 1 is $\operatorname{Re} W(z) > \beta$, $z \in U$, it contradicts the condition $f(z) \in M(\alpha, \beta)$ for $1 > \beta \geq 0$

COROLLARY 4. Theorem B is true, that is,

$$M(\alpha) \subset S^* \quad \text{for any real number } \alpha, \tag{3 3}$$

$$M(\alpha) \subset K \subset S^* \quad \text{for} \quad \alpha \geq 1. \tag{3 4}$$

PROOF. The fact (3.3) is a direct result of Corollary 3 We will show (3 4) Since $S^*(\gamma_1) \subset S^*(\gamma_2)$ for $0 \leq \gamma_2 \leq \gamma_1 < 1$, we have $M(\alpha) \subset S^*(\gamma) \subset S^*\left(\frac{1}{2}\right) \subset S^*$ for $\alpha \geq 1$ in (3 1) of Corollary 1 Therefore,

$$\operatorname{Re}\left\{\alpha\cdot\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\alpha)\frac{zf'(z)}{f(z)}\right\}>0,\quad z\in U \tag{3 5}$$

is written such as

$$\alpha\cdot\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>(\alpha-1)\operatorname{Re}\frac{zf'(z)}{f(z)}\geq 0.$$

This shows that $M(\alpha)\subset K$ for $\alpha\geq 1$ □

As an application of Theorem 2 we have the following:

COROLLARY 5. It holds that $N(\alpha,\beta)\subset K\subset S^*\left(\frac{1}{2}\right)$ for $\frac{1-\alpha}{2}\geq\beta>1$ and $\alpha<-1$

PROOF. We put $\gamma=\frac{1}{2}$ in (2 5) or (2 6) Then we obtain $\frac{1-\alpha}{2}\geq\beta>1$ and $\alpha<-1$ First this shows $N(\alpha,\beta)\subset S^*\left(\frac{1}{2}\right)$ Next the inequality (1 5)

$$\operatorname{Re}\left\{\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\alpha)\frac{zf'(z)}{f(z)}\right\}<\beta,\quad z\in U$$

implies that

$$\alpha\cdot\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)<\beta-(1-\alpha)\operatorname{Re}\frac{zf'(z)}{f(z)}<(1-\alpha)\left\{\frac{1}{2}-\operatorname{Re}\frac{zf'(z)}{f(z)}\right\}<0.$$

Thus we obtain $\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>0,\ z\in U$ □

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