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EXPONENTIAL STABILITY FOR ABSTRACT LINEAR AUTONOMOUS FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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ABSTRACT. Based on our preceding paper, this note is concerned with the exponential stability of the solution semigroup for the abstract linear autonomous functional differential equation

$$\dot{x}(t) = L(x_t) \tag{$*$}$$

where L is a continuous linear operator on some abstract phase space B into a Banach space E. We prove that the solution semigroup of $(*)$ is exponentially stable if and only if the fundamental operator $(*)$ is exponentially stable and the phase space B is an exponentially fading memory space.

KEY WORDS AND PHRASES: Exponentially stable, solution semigroup, abstract linear autonomous functional differential equation, fundamental operator, fading memory space.
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1. INTRODUCTION

Let E be a Banach space. For an E-valued function x defined on $(-\infty, a]$, by x_t we denote the function $x_t(\theta) = x(t+\theta)$, $-\infty < \theta \leq 0$. We consider the abstract linear autonomous functional differential equation with infinite delay

$$\begin{cases} \dot{x}(t) = L(x_t), & t > 0, \\ x(\theta) = \phi(\theta), & \theta \in (-\infty, 0] \end{cases} \tag{1 1}$$

where L is a continuous linear operator on an abstract phase space B into E.

In [1], we have discussed some problems on (1.1), and have given a definition of the fundamental operator for (1.1) corresponding to the fundamental matrix [2,3] of (1 1) in the case of finite-dimensional space, and some sufficient and necessary conditions of the fundamental operator of (1.1) being exponentially stable. In the present note, we investigate the exponential stability of the solution semigroup of (1.1) and set up a sufficient and necessary condition for it. It is shown that the solution semigroup of (1.1) is exponentially stable if and only if the fundamental operator of (1.1) is exponentially stable and the phase space B is an exponentially fading memory space.

2. MAIN RESULTS

Let $U(Y)$ be the set of all bounded linear operators from space Y to Y. $\chi_{[u,v]}(\cdot)$ denotes the characteristic function of interval $[u, v]$. Suppose B is a Banach space of E-valued functions on $(-\infty, 0]$ with a norm $\|\cdot\|_B$ having the following properties:

(H1). If $x : (-\infty, a] \to E$, $a > \sigma$, is continuous on $[\sigma, a]$ and $x_0 \in B$, then x_t is continuous in $t \in [\sigma, a]$.

(H2). $\|\phi(0)\| \le K\|\phi\|_B$ for all ϕ in B and some constant K.

(H3). $\|\phi\|_B \le \|\phi\|_\beta + \|\phi\|_{(\beta)}$ for any $\beta > 0$, $\phi \in B$, where $\|\phi\|_\beta = \inf\{\|\psi\|_B : \psi \in B$ and $\psi(\theta) = \phi(\theta)$ for $\theta \in (-\infty, -\beta]\}$, $\|\phi\|_{(\beta)} = \inf\{\|\psi\|_B : \psi \in B$ and $\psi(\theta) = \phi(\theta)$ for $\theta \in [-\beta, 0]\}$.

(H4). $\|\phi\|_\beta \le K_1 \sup\{\|\phi(\theta)\|, -\beta \le \theta \le 0\}$ for any $\phi \in B$ and some constant K_1; $\sup\limits_{\|\phi\|_B=1} \|\phi\|_\beta$ is a locally bounded function of $\beta \ge 0$.

(H5). $\|x(\cdot)\|_B \le M_1\|y(\cdot)\|_B$ for any $x, y \in B$ with $\|x(\theta)\| \le \|y(\theta)\|$ for $\theta \in (-\infty, 0]$ and some constant M_1.

(H6). $\chi_{[-t,0]}(\cdot)b$, $\chi_{(-t,0]}(\cdot)b \in B$ for any $t \ge 0$ and $b \in E$: $\|\chi_{[0,0]}(\cdot)b\|_B \le M\|b\|$ for $b \in E$ and some constant M.

B is called an *admissible phase space*. (H4) and (H5) stem from (α_3), (β_1) and (β_2) in [4]. It is clear that [1, H(3)] follows from (H4) and (H5). A typical example of the space B could be found in [1]. It is known that for each $\phi \in B$, the solution $x(t) = x(t, \phi)$ of (1.1) exists uniquely for $t \in [0, \infty)$ For every $\phi \in B$, $t \in [0, \infty)$, define

$$(T(t)\phi)(\theta) = x_t(\theta) = x_t(\theta, \phi) = \begin{cases} \phi(0) + \displaystyle\int_0^{t+\theta} L(x_s)ds, & t+\theta > 0, \\ \phi(t+\theta), & t+\theta \le 0, \end{cases} \quad \theta \in (-\infty, 0].$$

$\{T(t)\}_{t\ge0}$ is called the solution semigroup of (1.1). If $L = 0$, we denote by $\{S(t)\}_{t\ge0}$ the solution semigroup, i.e.,

$$(S(t)\phi)(0) = \begin{cases} \phi(0), & t+\theta > 0, \\ \phi(t+\theta), & t+\theta \le 0, \end{cases} \quad t \in [0, \infty),\ \phi \in B,\ \theta \in (-\infty, 0].$$

It is also called a translation semigroup. By virtue of (H1), $\{T(t)\}_{t\ge0}$ and $\{S(t)\}_{t\ge0}$ are C_0 semigroups on B. Set

$$(S_0(t)\phi)(\theta) = \begin{cases} 0, & t+\theta > 0, \\ \phi(t+\theta), & t+\theta \le 0, \end{cases} \quad t \in [0, \infty),\ \phi \in B,\ \theta \in (-\infty, 0].$$

An admissible phase space B is called to be an exponentially fading memory space, if there are constants $C, \alpha > 0$ such that $\|S_0(t)\phi\|_t \le Ce^{-\alpha t}\|\phi\|_B$ for any $\phi \in B$, $t > 0$. By (H1) and (H6), $S_0(t)\phi \in B(t > 0)$. Therefore, B is an exponentially fading memory space if and only if $\|S_0(t)\phi\|_B \le Ce^{-\alpha t}\|\phi\|_B$ for any $\phi \in B$, $t > 0$ and some positive constants C, α, since by (H5),

$$\|S_0(t)\phi\|_t \le \|S_0(t)\phi\|_B \le M_1\|S_0(t)\phi\|_t \quad \text{for any} \quad \phi \in B,\ t > 0. \tag{2.1}$$

LEMMA 1. Let $x(t, \phi)$ be the solution of (1.1). Then

$$x(t, \phi) = \begin{cases} X(t)\phi(0) + \displaystyle\int_0^t X(t-\tau)L(S_0(\tau)\phi)d\tau, & t > 0, \\ \phi(t), & t \le 0, \end{cases} \tag{2.2}$$

or for $\theta \in (-\infty, 0]$,

$$(T(t)\phi)(\theta) = \begin{cases} X(t+\theta)\phi(0) + \int_0^{t+\theta} X(t+\theta-\tau)L(S_0(\tau)\phi)d\tau, & t+\theta > 0, \\ \phi(t+\theta), & t+\theta \le 0, \end{cases}$$
$$= (S_0(t)\phi)(\theta) + X(t+\theta)\phi(0) + \int_0^{t+\theta} X(t+\theta-\tau)L(S_0(\tau)\phi)d\tau, \qquad (2.3)$$

where $X(t)$ is the fundamental operator of (1.1) (see [1]).

PROOF. Let

$$\|T(t)\phi\|_B, \quad \|S(t)\phi\|_B \le Me^{\eta t}, \quad t \ge 0, \quad \phi \in B, \qquad (2.4)$$

where $M > 0$, $\eta > \omega$ (the constant in [1, Lemma 2.2]) are constants. For $\phi \in B$, set

$$y(t) = y(t,\phi) = \begin{cases} 0, & t > 0 \\ \phi(t), & t \le 0, \end{cases}$$

and $z(t) = z(t,\phi) = x(t,\phi) - y(t,\phi)$, where $x(t,\phi)$ is the solution of (1.1). Then we have

$$\frac{dx}{dt} = L(z_t + S_0(t)\phi) = L(z_t) + L(S_0(t)\phi), \quad t > 0. \qquad (2.5)$$

By virtue of (H5) and (2.4), we obtain

$$\|z_t\|_B, \quad \|S_0(t)\phi\|_B \le M_1 M e^{\eta t}, \quad t > 0, \quad \phi \in B. \qquad (2.6)$$

Moreover, according to (H2) or (2.5),

$$\|z(t)\| < M_2 e^{\eta t}, \quad t > 0, \qquad (2.7)$$

where $M_2 > 0$ is a constant. Thus, taking Laplace transform on two sides of the equality (2.5), we have that for $\operatorname{Re}\lambda > \eta$,

$$-\phi(0) + \lambda\hat{z}(\lambda) = L(e^{\lambda\cdot})z(\lambda) + \int_0^\infty e^{-\lambda t}L(S_0(t)\phi)dt,$$

that is

$$\hat{z}(\lambda) = \left(\lambda I - L(e^{\lambda\cdot})\right)^{-1}\phi(0) + \int_0^\infty e^{-\lambda t}\left(\lambda I - L(e^{\lambda\cdot})\right)^{-1}L(S_0(t)\phi)dt,$$

where $\hat{z}(\lambda)$ is a Laplace transform of z. Accordingly, the formula (2.2) follows from the definition of $X(t)$, the convolution property and the uniqueness property of Laplace transform.

THEOREM 1. The solution semigroup $\{T(t)\}_{t\ge 0}$ of (1.1) is exponentially stable if and only if the fundamental operator $X(t)$ of (1.1) is exponentially stable and the phase space B is an exponentially fading memory phase.

PROOF. Necessity. Let

$$\|T(t)\phi\|_B \le M_0 e^{-\sigma t}\|\phi\|_B \quad \text{for} \quad t \ge 0, \ \phi \in B, \qquad (2.8)$$

where M_0, $\sigma > 0$ are constants. Then, by (H5)

$$\|S_0(t)\phi\|_t = \|T_0(t)\phi\|_t \le \|T(t)\phi\|_B \le M_0 e^{-\sigma\tau}\|\phi\|, \ t \ge 0, \ \phi \in B, \qquad (2.9)$$

that is, B is an exponentially fading memory space.

According to [1, Lemma 3.1], for every $b \in E$, $X(t)b$ satisfies

$$\begin{cases} \dot{X}(t)b = L(X_t b), & t > 0 \\ X(\theta)b = \chi_{[0,0]}(\theta)b, & \theta \in (-\infty, 0). \end{cases}$$

Therefore,

$$(T(t)\chi_{[0,0]}(\cdot)b)(\theta)|_{\theta=0} = b + \int_0^t L(X_\tau b)d\lambda = X(t)b,\ b \in E.$$

Consequently, it follows from (H2), (2.8) and (H6) that for any $b \in E$,

$$\|X(t)b\| \le K\|T(t)\chi_{[0,0]}(\cdot)b\| \le KM_0Me^{-\sigma t}\|b\|,\quad t \ge 0.$$

This shows that $X(t)$ is exponentially stable.

Sufficiency. Obviously, we can suppose

$$\|X(t)\| \le M_0e^{-\sigma t}(t \ge 0),$$
$$\|S_0(t)\phi\|_B \le M_0e^{-\sigma t}\|\phi\|_B\ \ (t > 0, \phi \in B).$$

Hence, by (H2),

$$\sup_{-\frac{t}{2}\le\theta<0} \|X(t+\theta)\phi(0)\| \le M_0e^{-\frac{\sigma}{2}t}\|\phi(0)\| \le M_0e^{-\frac{\sigma}{2}t}\|\phi\|_B. \tag{2.10}$$

Moreover, for $\theta \in \left[-\frac{t}{2}, 0\right]$,

$$\left\|\int_0^{t+\theta} X(t+\theta-\tau)L(S_0(\tau)\phi)d\tau\right\| \le \int_0^{t+\theta} M_0e^{-\sigma(t+\theta-\tau)}\|L\|M_0e^{-\sigma\tau}\|\phi\|_Bd\tau$$
$$\le M_0^2\|L\|(t+\theta)e^{-\sigma(t+\theta)}\|\phi\|_B.$$

Accordingly,

$$\sup_{-\frac{t}{2}\le\theta\le0}\left\|\int_0^{t+\theta} X(t+\theta-\tau)L(S_0(\tau)\phi)d\tau\right\| \le \frac{1}{2}M_0^2\|L\|te^{-\frac{\sigma}{2}t}\|\phi\|_B. \tag{2.11}$$

(H4), Lemma 1, (2.10) and (2.11) imply that

$$\|T(t)\phi\|_{(\frac{t}{2})} \le K_1 \sup_{-\frac{t}{2}\le\theta\le0} \|(T(\tau)\phi)(\theta)\| \le K_1 \sup_{-\frac{t}{2}\le\theta\le0} \|X(t+\theta)\phi(0)\| +$$
$$K_1 \sup_{-\frac{1}{2}\le\theta\le0}\left\|\int_0^{t+\theta} X(t+\theta-\tau)L(S_0(\tau)\phi)d\tau\right\| \le \left(K_1M_0 + \frac{1}{2}M_0^2\|L\|t\right)e^{-\frac{\sigma}{2}t}\|\phi\|_B.$$

As a consequence, there is a constant K_2 such that

$$\|T(t)\phi\|_{(\frac{t}{2})} \le K_2e^{-\frac{\sigma}{4}t}\|\phi\|_B, \tag{2.12}$$

taking into account $\lim_{t\to\infty} te^{-\frac{\sigma}{4}t} = 0$.

On the other hand,

$$\|T(t)\phi\|_{\frac{t}{2}} = \left\|T\left(\frac{t}{2}\right)\left(T\left(\frac{t}{2}\right)\right)\phi\right\|_{\frac{t}{2}} = \left\|S_0\left(\frac{t}{2}\right)\left(T\left(\frac{t}{2}\right)\phi\right)\right\|_{\frac{t}{2}}$$
$$\le \left\|S_0\left(\frac{t}{2}\right)\left(T\left(\frac{t}{2}\right)\phi\right)\right\| \le M_0e^{-\frac{\sigma}{2}t}\left\|T\left(\frac{t}{2}\right)\phi\right\|_B. \tag{2.13}$$

Therefore, it follows from (H4), (2.12) and (2.13) that

$$\|T(t)\phi\|_B \le \|T(t)\phi\|_{(\frac{t}{2})} + \|T(t)\phi\|_{\frac{t}{2}} \le K_2e^{-\frac{\sigma}{4}t}\|\phi\|_B + M_0e^{-\frac{\sigma}{4}t}\left\|T\left(\frac{t}{2}\right)\phi\right\|_B.$$

Consequently,

$$\|T(t)\| \le Ce^{-\frac{\sigma}{4}t}\left(1 + \left\|T\left(\frac{t}{2}\right)\right\|\right), \tag{2.14}$$

where $C = K_2 + M_0$.

We say that $\|T(t)\|$ is bounded on $[0, \infty]$. In fact, if it is false, then as shown in [5],

$$\omega_L = \inf_{t>0} \frac{\ln\|T(t)\|}{t} \geq 0, \tag{2.15}$$

that is, $\|T(t)\| \geq 1$. Hence, by (2 14),

$$\begin{aligned} \frac{\ln\|T(t)\|}{t} &\leq -\frac{\sigma}{4} + \frac{\ln C\left(1 + \|T(\frac{t}{2})\|^{-1}\right)}{t} + \frac{\ln\|T(\frac{t}{2})\|}{t} \\ &\leq -\frac{\sigma}{4} + \frac{\ln 2C}{t} + \frac{1}{2}\frac{\ln\|T(\frac{t}{2})\|}{\frac{t}{2}}. \end{aligned}$$

Letting $t \to \infty$, we obtain

$$\omega_L \leq -\frac{\sigma}{4} + \frac{1}{2}\omega_L,$$

i.e., $\omega_L \leq -\frac{\sigma}{2}$, which contradicts (2.15). So, $\|T(t)\|$ is bounded on $[0, \infty)$. Accordingly, (2.14) implies that $T(t)$ is exponentially stable.

REMARK 1. (i) If $L(S_0(t)\phi)$ is well defined for each $t > 0$, $\phi \in B$, and

$$L\left(\int_0^\infty e^{-\lambda t} S_0(t)\phi dt\right) = \int_0^\infty e^{-\lambda t} L(S_0(t)\phi) dt$$

for $Re\,\lambda > \omega$ (ω is a constant), then Lemma 1 still holds when the hypothesis (H6) on B is taken off

(ii) If the assumptions of (i) hold, $\|L(S_0(t)\phi)\| \leq M\|S_0(t)\phi\|_t$ for any $t > 0$, $\phi \in B$ and some constant $M > 0$, and E is a finite-dimensional space, then Theorem 1 is still true when the hypothesis (H6) on B is taken off.

PROOF. (i) and the "sufficiency" of (ii) are clearly true according to the proof of Lemma 1 and that of the "sufficiency" of Theorem 1 respectively.

Let us look at the "Necessity" of (ii) now. Obviously, by (2.9), B is an exponentially fading memory space. From (2.8), $\omega_L = \inf_{t>0} \frac{\ln\|T(t)\|}{t} \leq -\sigma$. Accordingly, for $Re\,\lambda > -\sigma$, $b \in E$, $e^{\lambda\theta}b \in B$ and $\det(\lambda - L(e^{\lambda \cdot})) \neq 0$, as shown in [3]. Taking a $\sigma_0 \in (0, \sigma)$, we have by (H5) that $\|e^{\lambda \cdot}b\|_B \leq M_1\|e^{-\sigma_0 \cdot}b\|_B$ and $\|L(e^{\lambda \cdot}b)\| \leq \|L\|M_1\|e^{-\sigma \cdot}b\|_B$ for $Re\,\lambda \geq -\sigma_0$. Thanks to the resonance theorem, we get $\sup\{\|L(e^{\lambda \cdot})\| : Re\,\lambda \geq -\sigma_0\} < \infty$. Therefore, $X(t)$ is exponentially stable by virtue of [1, Theorem 3.5].

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REFERENCES

[1] LIANG, J. and XIAO, T.J., Functional differential equations with infinite delay in Banach spaces, *Inter. J. Math. & Math. Sci.* **14** (1991), 497-508.

[2] CORDUNEANU, C. and LAKSHMIKANTHAM, V., Equations with unbounded delay: A survey, *Nonlinear Analysis Theory Method and Application* **4** (1980), 831-877.

[3] NAITO, T., On linear autonomous retarded equations with an abstract phase space for infinite delay, *J. Diff. Eqns.* **33** (1979), 74-91.

[4] HALE, J.K and KATO, J., Phase space for retarded equations with infinite delay, *Funcı. Ekvaciaj.* **21** (1978), 11-41.

[5] YOSIDA, K., *Functional Analysıs*, Springer-Verlag, Berlin, Heidelberg, 1965.