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RESEARCH NOTES

# A SUBSET OF METRIC PRESERVING FUNCTIONS

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**ABSTRACT.** In this paper we define a subset of metric preserving functions and give some examples and a characterization of this subset.

**KEY WORDS AND PHRASES:** Metrics, metric preserving functions, derivatives.
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## 1. INTRODUCTION

We call a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ a *metric preserving function* if and only if $f(\rho) : M \times M \to \mathbb{R}^+$ is a metric for every metric $\rho : M \times M \to \mathbb{R}^+$, where $(M, \rho)$ is an arbitrary metric space and $\mathbb{R}^+$ denotes the nonnegative reals. We will denote the collection of metric preserving functions by $\mathcal{M}$. There are many papers out there which deal with these functions (see the references). Of particular interest is the derivative of metric preserving functions. In [1] J. Boršik and J. Doboš show that if $f \in \mathcal{M}$ is differentiable then $|f'(x)| \leq f'(0)$. J. Doboš and Z. Piotrowski in [2] construct two examples concerning differentiation and metric preserving functions. The first $f \in \mathcal{M}$ is continuous and nowhere differentiable. The other is metric preserving, differentiable and the derivative is infinite exactly on $\{0\} \cup 2^{-n}$, $n = 1, 2, 3, \ldots$ . In [9] this author answers a question of Doboš and Piotrowski by showing how for any measure zero, $\mathcal{G}_\delta$ set in $[0, \infty)$ there is a continuous metric preserving function whose derivative is infinite on that set union zero.

The subset of metric preserving functions we wish to consider is defined below.

**DEFINITION.** Let $f \in \mathcal{M}$ be differentiable on $(0, \infty)$. Define $g(x)$ as

$$g(x) = \begin{cases} f'(x) & x \in (0, \infty) \\ 0 & x = 0 \end{cases} \tag{1.1}$$

We say $f \in \mathcal{D}$ if and only if $f, g \in \mathcal{M}$.

The purpose of this paper is to give examples of these types of functions and to characterize the type of $f$ which can be in $\mathcal{D}$.

## 2. MAIN RESULTS

We note here that the set $\mathcal{D}$ is nonempty. It is easy to see that $\mathcal{D}$ contains all functions of the form $f(x) = kx$, $k > 0$. A natural question to then ask is if it is possible that there are functions $f$ such that $g$ defined above is continuous at the origin (which is not that case for $f(x) = kx$). The answer is no and is given in the following theorem.

**THEOREM 1.** If $f$ is differentiable on $[0, \infty)$ and metric preserving $f'(x)$ is not a metric preserving function.

**PROOF.** If $f' \in \mathcal{M}$ then $f'(0)$ would have to be zero and $f' > 0$ on $(0, \infty)$ implies there must be some $[0, \epsilon)$ where $f$ must be strictly convex. Then $f \notin \mathcal{M}$ from Prop. 10 in [1].

Nor can we go in the opposite direction and assume that if $g$ is metric preserving its integral will also be metric preserving.

**EXAMPLE.** There exists a metric preserving function $g$ whose integral, $\int_0^x g(t), dt$, is not also metric preserving.

**PROOF.** Let $g(x) = 1 - e^{-x}$. Then $\int_0^x 1 - e^{-t} dt$ is strictly convex in a neighborhood of the origin.

Note that $g(x) = 2x$ would also serve in the example above. While both are continuous, $1 - e^{-x}$ has the added strength of being bounded. We now can look at some properties of these functions in $\mathcal{D}$.

**THEOREM 2.** If $f \in \mathcal{D}$, $f$ is nondecreasing.

**PROOF.** This is a consequence of the fact that the function $g(x)$ must be greater than zero since $g$ is metric preserving.

**LEMMA.** Let $f \in \mathcal{M}$ and $\limsup_{x \to 0^+} f(x) = a$. Then for all $x \in [0, \infty)$, $f(x) \geq a/2$.

**PROOF.** This is a property of $f$ being metric preserving. See Corollary 1 in [1].

**THEOREM 3.** Let $f(x) = x^k$. Only $f \in \mathcal{D}$ if and only if $k = 1$.

**PROOF.**

If $k > 1$ then $f \notin \mathcal{M}$ since $f$ would be strictly convex around the origin.

If $k \in (0, 1)$ then $g$ violates the lemma above.

If $k = 0$ then $g \notin \mathcal{M}$ since $g$ would be identically zero.

If $k < 0$ then $f$ violates the lemma above.

In order to characterize functions in the set $\mathcal{D}$ we need the notion of a triangle triplet. The 3-tuple $(a, b, c) \in (\mathbb{R}^+)^3$ is called a triangle triplet if $a \leq b + c$, $b \leq a + c$, and $c \leq a + b$. This is another way to determine if a function is metric preserving (see F. Terpe [8]). A function $f$ is a metric preserving function if and only if $f(0) = 0$ and $(f(a), f(b), f(c))$ is a triangle triplet whenever $(a, b, c)$ is one. This gives us a way to describe these functions in $\mathcal{D}$.

**THEOREM 4.** Let $g(x) : \mathbb{R}^+ \to \mathbb{R}^+$ be a function satisfying

$$\forall a > 0 \int_0^a g(x)dx \geq \int_b^c g(x)dx \quad \text{where} \quad c - b = a. \tag{2.1}$$

If there exists an $A > 0$ such that

$$A \leq N + Mg(x) \leq 2A, \tag{2.2}$$

then both $G(x) = \begin{cases} N + Mg(x) & x > 0 \\ 0 & x = 0 \end{cases}$ and $F(x) = \int_0^x G(t)dt$ are in $\mathcal{M}$.

**PROOF.** The condition (2.2) gives us $G(x)$ is metric preserving (Proposition 3 in [1]). Condition (2.1) assures that $F(x)$ will satisfy the triangle triplet condition. Assume $a < b < c$. Then $F(a) \leq F(b) + F(c)$ and $F(b) \leq F(a) + F(c)$ are automatic. Lastly,

$$F(c) = F(b) + \int_b^c G(t)dt \leq F(b) + \int_0^a G(t)dt = F(a) + F(b). \tag{2.3}$$

This describes such examples in $\mathcal{D}$ using $1 + e^{-x}$, $3 + \frac{1}{2}\cos(1/x)$, and $3 + e^{-x}\cos x$ for $N + Mg(x)$.

To close we note that this gives another way to create metric preserving functions.

**COROLLARY.** If $g(x)$ meets condition (2.1) and $0 \leq g(x)$ almost everywhere then $g(x)$ need not be in $\mathcal{M}$, but $\int_0^x g(t)d\lambda$ is in $\mathcal{M}$ where $\lambda$ denotes Lebesgue measure.

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