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OBSERVATIONS ON A VARIANT OF COMPATIBILITY

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ABSTRACT. We consider a variation of the concept of compatible maps introduced by Hicks and Saliga [1], and obtain generalizations of results by Hicks and Saliga and others.

KEY WORD AND PHRASES: compatible *with*, symmetric, d-complete topological space, fixed point.
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1. **INTRODUCTION.** The concept of compatibility was introduced in 1986 in [2]. Self maps f and g of a metric space (X, d) are *compatible* (or a *compatible pair)* iff $d(fgx_n, gfx_n) \to 0$ when $\{x_n\}$ is a sequence in X such that $fx_n, gx_n \to p \in X$. Since then this concept has been used extensively in published fixed point research, and a variety of variations and generalizations of the concept have appeared (See, e.g., [3,4,5]). Most of these variations were defined in the setting of metric spaces or probabilistic metric spaces.

Recently Hicks and Saliga introduced an interesting variant of compatibility for functions on a topological space (X, t) paired with a distance function $d: XxX \to [0, \infty)$ having the property that d(x, y) = 0 iff x = y. The spaces X are said to be *d-complete* iff any sequence $\{x_n\}$ for which $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$ converges to a point p of X. The distance function d is said to be a *symmetric* iff d(x, y) = d(y, x) for $x,y \in X$, and a symmetric d is a *semi-metric* iff $S(x, \epsilon) = \{y \in X: d(x, y) < \epsilon\}$ is a neighborhood of x for all $x \in X$ and for any $\epsilon>0$. A map $S:D(\subseteq X) \to X$ is w-continuous at $p \in D$ iff whenever $\{x_n\}$ is a sequence in D such that $x_n \to p \in D$, then $Sx_n \to Sp$. For further discussion of d-complete topological spaces and symmetrics/semi-metrics see [6].

In this paper we shall focus on the Hicks and Saliga compatibility concept. However, the concept was introduced in [1] for functions $S,T: D \to X$ $(D \subseteq X)$ and used on subsets C of the domains D, even though the precise meaning of this compatibility on subsets C was not made clear by the definition. We shall therefore begin by defining what we mean by S being compatible *with* T so as to (hopefully) preserve the sense intended by Hicks and Saliga.

DEFINITION 1.1. Let X be a topological space, let $C \subseteq D \subseteq X$, and let $T,S:D \to X$. S is *compatible with T on C* iff whenever $\{x_n\}$ is a sequence in C such that $\{Sx_n\}$ is in D and $Tx_n, Sx_n \to p \in D$, then $TSx_n \to Sp$. If C= D, we say *S is compatible with T*. It follows that if C = D = X, *S is compatible with T* iff whenever $\{x_n\}$ is a sequence in X such that $Sx_n, Tx_n \to p \in X$, then $TSx_n \to Sp$.

As we shall see, the relation, "compatible with ", is not necessarily commutative, whereas the concept of compatible pairs {S, T} introduced in [2] is. Although the metric space definition of compatible pairs extends naturally to a topological space having a symmetric, in this paper we are more interested in a property shared by these two compatibility concepts, namely, weak compatibility. Weak compatibility was defined in [7] for semi-metric spaces. We now define it for any set X.

DEFINITION 1.2. Let X be any set, let $C \subseteq D \subseteq X$, and let $T,S: D \to X$. S and T are *weakly compatible on C* iff ($x \in C$ and $Tx = Sx \in D$) $\Rightarrow$ ($STx = TSx$). If C = D, we say that S and T are weakly compatible – or the pair {S,T} is weakly compatible.

In the following N will denote the set of positive integers, and for $k \in N$, N_k is the set of all $n \in N$ such that $n \leq k$. If S is a map, we shall write Sx for S(x) when convenient and the meaning is clear. Moreover, we require that the topological spaces (X, t) be Hausdorff (which we designate "T_2") to ensure that converging sequences have unique limits (See example 2.2 in [7]).

2. THEOREMS AND RESULTS.

PROPOSITION 2.1. Let (X, t) be a T_2 topological space and let $C \subseteq D \subseteq X$. Let $S,T:D \to X$ and suppose that S is compatible *with* T on C.

1. If $x \in C$ and $Sx = Tx \in D$, then $T^2x = TSx = STx = S^2x$.

2. If $\{x_n\}$ is a sequence in C such that $Sx_n \in D$ for $n \in N$ and $Sx_n, Tx_n \to p \in D$, and if T is w-continuous at p, then Sp = Tp.

3. If C=D=X and both S and T are w-continuous, then T is compatible *with* S.

PROOF. Suppose that $x \in C$ and $Sx = Tx \in D$. Let $x_n = x$ for $n \in N$. Then $Sx_n, Tx_n \to Tx \in D$, and $TSx = \lim_{n \to \infty} TSx_n = STx$ since S is compatible *with* T on C; i.e., the conclusion 1. holds. To see that 2. is true, note that $TSx_n \to Sp$ since S is compatible *with* T on C. But since T is w-continuous at p and $\{Sx_n\}$ is a sequence in D convergent to p, $TSx_n \to Tp$. But (X, t) is T_2 and therefore limits of sequences are unique; i.e., Tp = Sp.

And to prove 3., suppose $Tx_n, Sx_n \to p \in X$. Since S is w-continuous, $STx_n \to Sp$. But Sp = Tp by 2. and therefore $STx_n \to Tp$; i.e., T is compatible *with* S. □

Note that Proposition 2.1.1 tells us that if S is compatible *with* T (on C), then the pair {S, T} is weakly compatible (on C), even though – as the next example shows – T is not compatible *with* S. The following example also shows us that the conclusion of Proposition 2.1.2 need not hold if S is compatible *with* T and T is <u>not</u> continuous at p, even though S is continuous at p.

EXAMPLE 2.1. Let X = [0, 1] with $d(x, y) = |x - y|$, let S = I (the identity map) and define $T:X \to X$ by $Tx = (2x+3)\, 8^{-1}$ if $x \in [0, \frac{1}{2})$ and Tx = 0 if $x \in [\frac{1}{2}, 1]$. .Now, as will be shown momentarily, I is compatible *with* (any) T and certainly is continuous. But if $x_n \to \frac{1}{2}$ from the left, $Tx_n \to \frac{1}{2}$, $ITx_n \to \frac{1}{2}$, whereas $T(\frac{1}{2}) = 0$; i.e., T is not compatible *with* I. And $T(\frac{1}{2}) = 0 \neq \frac{1}{2} = I(\frac{1}{2})$, so the conclusion 2 in Proposition 2.1 does not hold.

In that which follows, we shall use the collapsing principle for series. Thus, if $\{a_k\}$ is a sequence of numbers, then $\sum_1^n (a_k - a_{k+1}) = a_1 - a_{n+1}$ for $n \in N$.

THEOREM 2.1. Let (X, t) be a d-complete T_2 topological space and let D be a closed subset of X. Let $S,T:D \to X$ where $S(D) \subseteq D \cap T(D)$. Suppose there is a map $\alpha:D \to [0, \infty)$ such that

$$d(Tx, Sx) \leq \alpha(Tx) - \alpha(Sx) \text{ for } x \in T^{-1}(D).$$

Then, if $x_o \in D$ $\exists$ sequences $\{x_n\},\{y_n\}$ in D such that $y_n = Sx_{n-1} = Tx_n$ for $n \in N$ and $y_n \to p \in D$. Moreover, if T is w-continuous at p and S is compatible *with* T on $T^{-1}(D)$, then Sp=Tp.

PROOF. Since $S(D) \subseteq T(X)$, given $x_o \in D$, we can choose $x_1 \in D$ such that $Tx_1 = Sx_o$. We can then choose $x_2 \in D$ such that $Tx_2 = Sx_1$. In general, given $x_i \in D$ for $i \in N_k$ such that $Tx_i = Sx_{i-1}$, we can choose $x_{k+1} \in D$ such that $Tx_{k+1} = Sx_k$. Thus, by induction, a sequence $\{y_n\}$ of the type cited in the statement of the theorem exists. Since $Tx_k = Sx_{k-1} \in D$ for $k \in N$, $x_k \in T^{-1}(D)$ for $k \in N$, so for $k \in N$ we can write:

$d(y_k, y_{k+1}) = d(Tx_k, Sx_k) \leq \alpha(Tx_k) - \alpha(Sx_k) = \alpha(y_k) - \alpha(y_{k+1})$, which by the collapsing principle implies

$$\sum_{k=1}^{n} d(y_k, y_{k+1}) = \alpha(y_1) - \alpha(y_{n+1}) \leq \alpha(y_1), \text{ for } n \in N.$$

Thus $\sum_{k=1}^{\infty} d(y_k, y_{k+1}) < \infty$. Therefore, $y_k \to p \in X$ since (X, t) is d-complete.

Consequently, $Sx_k, Tx_k \to p$. But $Sx_k \in D$ for $k \in N$ and D is closed, so that $p \in D$. Moreover, if S is compatible *with* T on $T^{-1}(D)$ and T is w-continuous at p, since $x_k \in T^{-1}(D)$ for $k \in N$, Proposition 2.1. (with $C = T^{-1}(D)$) implies that $Tp = Sp$. □

Example 2.1 shows us that even though the identity map I on a space X is compatible *with* any map on X, ∃ maps S on X not compatible *with* I. The following proposition says that nice things happen when a function S is compatible *with* I.

PROPOSITION 2.2. Let X be a T_2 topological space and let $D \subseteq X$. Suppose $S:D \to X$ and I is the identity map. Then I is compatible *with* S. Moreover, if S is compatible *with* I and $\{Sx_n\}$ and $\{x_n\}$ are sequences in D which converge to $p \in D$, then $Sp = p$. And if S is w-continuous, then S is compatible *with* I.

PROOF. To see that I is compatible *with* S, let $\{x_n\}$ be a sequence in D such that $Ix_n = x_n$, $Sx_n \to p \in D$. Then $SIx_n = Sx_n \to p = Ip$, so I is (trivially) compatible *with* S.

If S is compatible *with* I and $\{x_n\}$, $\{Sx_n\}$ are sequences in D which converge to $p \in D$, then $p= Sp$ by Proposition 2.1 (with $T = I$).

Now suppose that S is w-continuous and let $\{x_n\}$, $\{Sx_n\}$ be sequences in D such that $Ix_n, Sx_n \to p \in D$. Then, since $Ix_n(= x_n) \to p \in D$, $Sx_n \to Sp$ by continuity. Thus, $Sx_n= ISx_n \to Sp$, and hence S is compatible *with* I. □

The last sentence in Proposition 2.2 prompts the question, "If $S:D \to X$ is compatible *with* I (the identity map), is S continuous on D?" The next example tells us that the answer is "no", even if $S:X \to X$.

EXAMPLE 2.2. Let $X = [0, 1]$ with the usual metric and define $S: X \to X$ by $Sx = 1$ if $x \in [0, \frac{1}{2}]$ and $Sx = 0$ if $x \in (\frac{1}{2}, 1]$. Now if $Sx_n \to p$, then $p \in \{0, 1\}$. But if $Ix_n(=x_n) \to 0$, $Sx_n \to 1$ and if $Ix_n \to 1$, $Sx_n \to 0$; i.e., there is no $p \in X$ such that $Ix_n, Sx_n \to p$. Thus, S is compatible *with* I vacuously, but S is certainly not w-continuous.

COROLLARY 2.1. Let (X, t) be a d-complete T_2 topological space and let $S:X \to X$. If there exists a map $\alpha: X \to [0, \infty)$ such that for $x \in X$

$$d(x, Sx) \leq \alpha(x) - \alpha(Sx),$$

then for any $x \in X$, $S^n(x) \to p$ for some $p = p_x \in X$. If S is compatible *with* the identity map, then $Sp = p$.

(To see that $p=p_x$ need not be unique, let $S = I$, the identity map.)

PROOF. If we let $T = I$, the identity map, in Theorem 2.1, then $y_n = x_n = Sx_{n-1}$ for $n \in N$. Thus, $y_n= S^n x_o$ for $n \in N$. Since $I(X) = X$, the conclusion follows. □

NOTE 2.1. Corollary 2.1 is the topological version of Caristi's Theorem [8] for complete metric spaces. Caristi required that α be lower semi-continuous, whereas we required that S be compatible *with* the identity map. Dien [10] noted – as Browder [9] had already known in 1975 – that for metric spaces, the lower semi-continuity requirement on α can be dropped by requiring that S be continuous. In view of Example 2.2, Dien's comment suggests that Corollary 2.1 is of interest.

COROLLARY 2.2. Let (X, t) be a d-complete T_2 topological space and let $T:D \to X$, where D is a closed subset of X. If $D \subseteq T(D)$ and if ∃ a map $\alpha: X \to [0, \infty)$ such that

$$d(Tx, x) \leq \alpha(Tx) - \alpha(x), \text{ for } x \in T^{-1}(D),$$

then any $x_o \in D$ determines a sequence $\{x_n\}$ with $x_{n-1} = Tx_n$ such that $x_n \to p \in D$. If T is w-continuous at p, then $Tp = p$.

PROOF. Let $S = I$, the identity map in Theorem 2.1., and note that I is compatible with T by Proposition 2.2. □

THEOREM 2.2. Let (X, t) be a d- complete Hausdorff topological space and let D be a closed subset of X. Suppose $S, T: D \to X$ and that $S(D) \subseteq D \cap T(D)$. If $\exists$ maps $\alpha, \beta: X \to [0,\infty)$ and $r \in (0, 1)$ such that

$$(*) \quad d(Sx, Sy) \leq r\, d(Tx, Ty) + (\alpha(Tx) - \alpha(Sx)) + (\beta(Ty) - \beta(Sy))$$

for $x,y \in T^{-1}(D)$, then for any $x_o \in D$, $\exists$ sequences $\{x_n\}$, $\{y_n\}$ such that $y_n = Tx_n = Sx_{n-1}$ for $n \in N$ and $y_n \to p = p_{x_o} \in D$. If S is compatible *with* T on $T^{-1}(D)$ and T is w-continuous at p, then $Tp = Sp = p$, and p is the only common fixed point of S and T.

PROOF. Let $x_o \in D$. As in the proof of Theorem 2.1, we construct the sequences $\{x_n\}$, $\{y_n\}$ in D such that $y_n = Tx_n = Sx_{n-1} \in D$ for $n \in N$. Then $x_k \in T^{-1}(D)$ for $k \in N$, so (*) implies for $k \in N$:

$$d(y_k, y_{k+1}) = d(Sx_{k-1}, Sx_k)$$
$$\leq r\, d(Tx_{k-1}, Tx_k) + (\alpha(Tx_{k-1}) - \alpha(Sx_{k-1})) + (\beta(Tx_k) - \beta(Sx_k)), \text{ or}$$
$$d(y_k, y_{k+1}) \leq r\, d(y_{k-1}, y_k) + (\alpha(y_{k-1}) - \alpha(y_k)) + (\beta(y_k) - \beta(y_{k+1})).$$

Then the collapsing principle yields:

$$\sum_{k=1}^{n+1} d(y_k, y_{k+1}) \leq r \sum_{k=1}^{n+1} d(y_{k-1}, y_k) + (\alpha(y_o) - \alpha(y_{n+1})) + (\beta(y_1) - \beta(y_{n+2})), \text{ or}$$
$$\sum_{k=1}^{n} d(y_k, y_{k+1}) + d(y_{n+1}, y_{n+2})$$
$$\leq r \sum_{k=1}^{n} d(y_k, y_{k+1}) + r\, d(y_o, y_1) + \alpha(y_o) + \beta(y_1).$$

We drop the second term in the left member of the last inequality above to obtain,

$$(1 - r) \sum_{k=1}^{n} d(y_k, y_{k+1}) \leq \alpha(y_o) + \beta(y_1) + r\, d(y_o, y_1) = M, \text{ a constant} \geq 0.$$

Thus, for $n \in N$: $\sum_{k=1}^{n} d(y_k, y_{k+1}) \leq M(1 - r)^{-1}$, a nonnegative real, since $0 < r < 1$. Therefore, $\sum_{k=1}^{\infty} d(y_k, y_{k+1}) < \infty$, and $\{y_k\}$ converges to $p \in X$ since X is d-complete. So $Tx_k, Sx_k \to p = p_{x_o}$. But $Sx_k \in D$ for $k \in N$ and D is closed, which implies that $p \in D$. By the above, $\{x_k\}$ is a sequence in $C = T^{-1}(D)$ such that $Sx_k \in D$ for $k \in N$, and $Tx_k, Sx_k \to p \in D$. Therefore, if S is compatible *with* T on $T^{-1}(D)$ and T is w-continuous at such a p, Proposition 2.1 (2). implies that $Sp = Tp$. But since $p \in D$, $Sp(=Tp) \in D$, so that $p \in T^{-1}(D)$. Thus Proposition 2.1(1) with $C = T^{-1}(D)$ tells us that

$$S^2p = STp = TSp = T^2p \tag{2.1}$$

Moreover, since $Tp = Sp \in D$ and $TSp = SSp \in D$, we know $p, Sp \in T^{-1}(D)$. So (*) implies

$$d(Sp, SSp) \leq r\, d(Tp, TSp) + (\alpha(Tp) - \alpha(Sp)) + (\beta(TSp) - \beta(STp)) = r\, d(Tp, TSp) + 0;$$

i.e., $d(Sp, SSp) \leq r\, d(Sp, SSp)$ by (2.1). Since $r \in (0, 1)$, and $d(x, y) = 0$ implies $x=y$, we have $Sp = SSP$. Then $Sp = SSp = TSp$ by (2.1), so that Sp is a common fixed point S and T. That Sp is the only common fixed point of S and T follows easily from (*). □

If we let α, β be identically 0 in Theorem 2.2, we obtain the sufficency portion of Theorem 3. in [1], with the assumption that the phrase , "g is compatible *with* f on $f^{-1}(C)$" in the statement of the Theorem 3. conforms to our definition. Note also that the argument given in the proof of Theorem 3. to prove the necessity portion could be used to obtain a necessary and sufficient condition that the T in Theorem 2.2 have a fixed point.

NOTE 2.2 In this paper, Q denotes a nondecreasing map $Q:[0,\infty) \rightarrow [0, \infty)$ such that $Q(t) < t$ for $t > 0$ and $\sum_{n=1}^{\infty} Q^n(t)$ converges for all t. Consequently, $Q(0) = 0$, and $d(x, y) \leq Q(d(x, y))$ implies that $x = y$.

Our final result appeals to the following lemma.

LEMMA 2.1. Let (X, t) be a d-complete topological space with d symmetric, and let $D \subseteq X$. Let $A,B,S,T: D \rightarrow X$, such that $A(D) \subseteq T(D)$ and $B(D) \subseteq S(D)$. If

(i) $d(Ax, By) \leq Q(m(x, y))$ for $x,y \in X$, where

$m(x, y) = \max \{ d(Ax, Sx), d(By, Ty), d(Sx, Ty) \}$,

then the sequences $\{y_n\}$ defined below in (ii) exist for any $x_o \in D$ and converge to a point $p \in X$.

(ii) $y_{2n} = Sx_{2n} = Bx_{2n-1}$, $y_{2n-1} = Ax_{2n-2} = Tx_{2n-1}$, and $x_n \in D$ for $n \in N \cup \{0\}$.

Lemma 2.1 is proved in [7]. It is proved for a semi-metric d, but is valid if d is a symmetric. In [7] the sequence $\{y_n\}$ is proven to be "d-Cauchy" which justifies the conclusion above that, with the above hypothesis of "d-completeness", $\{y_n\}$ converges to a point $p \in X$.

THEOREM 2.3. Let (X, t) be a d-complete Hausdorff topological space with d symmetric and let D be a closed subset of X. Suppose $A,B,S,T:D \rightarrow X$, $A(D) \subseteq D \cap T(X)$ and $B(D) \subseteq D \cap S(D)$. If S and T are w-continuous, if (i) in Lemma 2.1 holds, and if A(B) is compatible *with* S(T), then A,B,S, and T have a unique common fixed point.

PROOF. Let $x_o \in D$ and let $\{y_n\}$, $\{x_n\}$ be sequences assured by Lemma 2.1, so that we have

$$y_n, Ax_{2n}, Bx_{2n-1}, Sx_{2n}, Tx_{2n-1} \rightarrow p \in X.$$

We know that $p \in D$ since $Ax_{2n} \in D$ for all n and D is closed. Since A is compatible with S and S is w-continuous, and since $Ax_{2n}, Sx_{2n} \rightarrow p \in D$, Proposition 2.1 implies

$$Ap = Sp, \text{ and } A^2p = ASp = SAp = S^2p. \text{ Similarly,} \tag{2.2}$$

$$Bp = Tp, \text{ and } B^2p = BTp = TBp = T^2p. \tag{2.3}$$

Then by property (i) (Lemma 2.1),

$$\begin{aligned} d(Ap, Bp) &\leq Q(\max\{ d(Ap, Sp), d(Bp, Tp), d(Sp, Tp)\}) \\ &\leq Q(\max\{ 0, 0, d(Sp, Tp)\}) = Q(d(Ap, Bp)), \text{ by (2.2) and (2.3).} \end{aligned}$$

Therefore, $Ap = Bp$ by Note 2.2, and hence

$$Ap = Bp = Sp = Tp. \tag{2.4}$$

Then (i) implies $d(Ap, BAp) \leq Q(\max\{ d(Ap, Sp), d(BAp, TAp), d(Sp, TAp)\})$, or

$$d(Ap, BAp) \leq Q(\max\{ 0, 0, d(Sp, TAp) \}) = Q(d(Ap, BAp)),$$

by (2.2), (2.3), and (2.4) above. Therefore, $Ap = BAp$. By symmetry we also have $Bp = ABp$. Therefore, the above equalities yield $Ap = AAp = BAp = SAp = TAp$; i.e., Ap is a common fixed point of A,B,S, and T. That $x = Ap$ is the only common fixed point of A,B,S and T follows from the contraction property (i). □

We now state a result described in the final paragraph in [11] which is a variation of their Theorem 5., page 793.

THEOREM 5B. [11] Let A, B, S, and T be mappings of a d-topological space (X,t) into itself satisfying the following conditions:

(a) $A(X) \subseteq T(X)$ and $B(X) \subseteq T(X)$,

(b) $d(Ax, By) \leq \phi(\max\{ d(Ax, Sx), d(By, Ty), d(Sx, Ty) \})$ for all $x,y \in X$,
where $\phi: [o, \infty) \rightarrow [0, \infty)$, $\phi(0) = 0$, ϕ is nondecreasing, ϕ is upper continuous, and $\phi(t) < t$ for $t > 0$,

(c) S and T are w-continuous,

(d) A, S and B,T satisfy the Hicks and Saliga definition of compatibility, and

(e) d is a continuous symmetric.

Then A, B, S, and T have a unique common fixed point, provided $\exists\, x_o, x_1 \in X$ such that $Ax_o = Tx_1$ and $\sum_{n=1}^{\infty}\phi^n(d(Ax_o, Bx_1)) < \infty$.

Our Theorem 2.3 has Theorem 5B as a corollary. In fact, Theorem 5B has the following restrictions in the hypothesis not required by Theorem 2.3. The contractive function ϕ is upper semi-continuous. A,S and B,T are compatible in the sense of "Hick's and Saliga's definition", which suggests that A(B) is compatible with S(T) and conversely. They also require that the symmetric d be continuous and that the domains of A,B,S, and T be X.

We should note that Harder and Saliga require that $\exists\, x_o, x_1 \in X$ such that $Ax_o = Tx_1$ and $\sum_{n=0}^{\infty}\phi^n(d(Ax_o, Bx_1)) < \infty$. Thus, their requirement for convergence for one t appears to be lighter than our requirement that $\sum_{n=0}^{\infty}Q^n(t) < \infty$ for all $t \in [0, \infty)$. However, the next result states that because of the requirements imposed on ϕ in (b) of the hypothesis, the inequality $\sum_{n=0}^{\infty}\phi^n(t) < \infty$ holds for all t if it holds for one $t \in (0,\infty)$.

PROPOSITION 2.3. Suppose that $\phi:[0, \infty) \to [0, \infty)$, ϕ is upper-semicontinuous, nondecreasing, and that $\phi(t) < t$ for $t > 0$. If $\sum_{n=1}^{\infty}\phi^n(t_o) < \infty$ for some $t_o > 0$, then $\sum_{n=1}^{\infty}\phi^n(t) < \infty$ for all $t > 0$.

PROOF. We first prove that

$$(1) \quad \lim_{n\to\infty}\phi^n(t) = 0 \text{ for any } t \in [0, \infty),$$

Since ϕ is nondecreasing, $\phi(0) \le \phi(t) < t$ for any $t > 0$; therefore $\phi(0) = 0$. So suppose that $t > 0$. Then $\phi(t) < t$, and $\phi^2(t) = \phi(\phi(t)) \le \phi(t)$, since ϕ is nondecreasing. In general, $0 \le \phi^{n+1}(t) \le \phi^n(t)$. Thus $\lim_{n\to\infty}\phi^n(t) = c^+ \ge 0$, where $\phi^n(t) \ge c$ for all $n \in N$. If $c > 0$, $\phi(c) < c$. Thus, since ϕ is upper semi-continuous, $\exists\, \delta > 0$ such that $\phi(t) < c$ for $t \in (c - \delta, c + \delta)$. Since $\phi^n(t) \to c$ as $n \to \infty$, $\phi^m(t) \in (c - \delta, c + \delta)$ for some m. Therefore, we have the contradiction $\phi^{m+1}(t) < c \le \phi^{m+1}(t)$, by the choice of δ.

Now suppose that $t_o > 0$ and $\sum_{n=1}^{\infty}\phi^n(t_o) < \infty$. Let $t \in (0, \infty)$. By (1), we can choose $m \in N$ such that $\phi^m(t) < t_o$, so $\phi^{m+1}(t) \le \phi(t_o)$, and in general we have $\phi^{m+n}(t) \le \phi^n(t_o)$ for $n \in N$, since ϕ is nondecreasing. But then,

$$\sum_{n=1}^{\infty}\phi^{m+n}(t) \le \sum_{n=1}^{\infty}\phi^n(t_o) = M < \infty, \text{ and therefore,}$$

$$\sum_{k=1}^{\infty}\phi^k(t) = \sum_{k=1}^{m}\phi^k(t) + \sum_{n=1}^{\infty}\phi^{m+n}(t) \le \sum_{k=1}^{m}\phi^k(t) + M < \infty. \ \square$$

3. RETROSPECT. In Definition 1.2 we defined the concept of a weakly compatible pair, and in Proposition 2.1 we proved in part 1. that "S compatible *with* T" implies that {S, T} is a weakly compatible pair. We close with an example which shows that even though S is not compatible *with* T and T is not compatible *with* S, {S, T} may be a weakly compatible pair. Moreover, since in this example S and T have a common fixed point and satisfy a contractive condition, $d(Sx, Sy) \le \frac{1}{2}\, d(Tx, Ty)$, we ask, to what extent can weak compatibility be used in lieu of the stronger forms of compatibility and still produce common fixed points? (A partial answer can be found in Theorems 3.1 and 3.2 of [7] in the context of semi-metric spaces.)

EXAMPLE. Let $X = [0, 2]$, $D = [0, 1]$, and $d(x, y) = (x - y)^2$ for $x,y \in X$. Let $Ax = (3/4) - (x/2)$ and $Sx = 2 - 3x$ for $x \in [0, \frac{1}{2}]$, and $Ax = \frac{3}{4}$ and $Sx = 0$ for $x \in (\frac{1}{2}, 1]$. Then $A,S: D \to X$ and $A(X) \subseteq S(X)$, and d is a semi-metric but not a metric (no triangle inequality). Now $\{A, S\}$ is weakly compatible since $Ax = Tx$ iff $x = \frac{1}{2}$, and $A(\frac{1}{2}) = S(\frac{1}{2}) = \frac{1}{2} = AS(\frac{1}{2}) = SA(\frac{1}{2})$. On the other hand consider $x_n = \frac{1}{2} - (\frac{1}{2})^n$ for $n>1$. Then $Ax_n, Sx_n \to \frac{1}{2}$, $SAx_n \to 0$ but $A(\frac{1}{2}) = \frac{1}{2}$ and $AS \to 3/4$ and $S(\frac{1}{2})=\frac{1}{2}$. Thus A is not compatible *with* S and S is not compatible *with* A. But $\frac{1}{2}$ is a common fixed point of A and S, and a quick check shows that $d(Ax, Ay) \le \frac{1}{4} d(Sx, Sy)$.

We conclude by observing that the expression (*) in our Theorem 2.2 was prompted by the analogous but more general expression (2.1) used by Dien in Theorem 2.1 [9]. The left member of Dien's contractive expression (2.1) was $d(Sx, Ty)$ and our expression (*) used $d(Sx,Sy)$. However, Dien's theorem is for metric spaces, and a check of his proof(s) reveals the central role of the triangle inequality, which we did not have for topological spaces with only a symmetric. Note also that the right member of (2.1) [9] contained an expression of the form

$$\sum_{i=1}^{n}[\phi_i(Ix) - \phi_i(Sx)] \tag{3.1}$$

where $\phi_i: X \to [0, \infty)$ for $i \in N_n$. But if we let $\alpha = \sum_{i=1}^{n}\phi_i$, then $\alpha: X \to [0, \infty)$ and (3.1) can be written $[\alpha(Ix) - \alpha(Sx)]$; i.e., no generality is gained by using n functions ϕ_i in lieu of one function α.

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