A CLOSER LOOK AT SOME NEW IDENTITIES

GEOFFREY B. CAMPBELL

Centre for Mathematics and its Applications
School of Mathematical Sciences
Australian National University
Canberra, ACT 0200, Australia

ABSTRACT.

We obtain infinite products related to the concept of visible from the origin point vectors. Among these is

$$\prod_{k=3}^{'}\left(1 - z^k\right)^{\varphi_3(k)/k} = \frac{1}{1-z}\exp\left(\frac{z^3}{2(1-z)^2} - \tfrac{1}{2}z - \frac{1}{2z(1-z)}\right), \qquad |z| < 1.$$

in which $\varphi_3(k)$ denotes for fixed k, the number of positive integer solutions of $(a,b,k) = 1$ where $a < b < k$, assuming (a,b,k) is the gcd(a,b,k)

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1.   INTRODUCTION.

In several of the author's recent papers (see Campbell [5-9]) a class of new elementary infinite product identities was introduced. These were given the name *vpv identities*, meaning *visible (from the origin) point vector identities*, due to the fact that they involved summation over so called visible lattice points in various dimensional spaces   For an introduction to the idea of visible lattice points see Apostol [2], where their distribution is calculated   The identities which occurred most frequently in Campbell [7] were often related to each other by vpv lattice sums dividing up space into radial regions from the origin These identities when grouped were called *companion identities*, because of their interdependence   We used the operators defined by

$$\sum_{k} \sum_{j} = \sum_{k=2}^{\infty} \sum_{j \in \phi_k} , \qquad \prod_{k} \prod_{j} = \prod_{k=2}^{\infty} \prod_{j \in \phi_k} , \qquad (1.1)$$

where ϕ_k is the set of positive integers less than and relatively prime to k. Whilst these operators are useful for the 2-D sums, as shown in the author's papers [5-6, 9], we will not require them for the higher dimensional sums considered here

Recently Mosseri [13], and Baake et al [3], have considered the so-called Optical Fourier Transform of the 2-D visible lattice points as an optical experiment. The subject of lattice point methods applied to physics and chemistry is also examined in Frankel et al [10], and Glasser and Zucker [11] Some of the methods of these papers are applicable to vpv sums and products. In Campbell [5-9] the following companion identities were given.

$$(1-y) \sum_{k} \sum_{j} (1 - x^j y^k)^{1/k} = \left(\frac{1-y}{1-xy} \right)^{1/(1-x)} , \qquad (x \neq 1) \qquad (1.2)$$

$$(1-xy) \sum_{k} \sum_{j} (1 - x^k y^j)^{1/j} = (1 - xy)^{1/(1-x)} , \qquad (1.3)$$

$$(1-y)(1-xy) \prod_{k} \prod_{j} (1 - x^j y^k)^{1/k} (1 - x^k y^j)^{1/j} = (1 - y)^{1/(1-x)} , \qquad (1.4)$$

valid respectively for $|xy|$ and $|y| < 1$, $|x|$ and $|xy| < 1$, $|x|$ and $|y| < 1$.

These led to ideas which link to several fields of research. For example, new products over the primes were presented in Campbell [6-7,9], new Jacobi theta function identities in [6]. Paper [6] also contained results on Dirichlet summations such as those connected with Ramanujan arithmetical functions, and was related to a method of Meinardus for obtaining asymptotic estimates of coefficients from infinite products (see Andrews [1], Hardy and Wright [11]). In [7], some further identities identities of this type were given and the existence of yet others was indicated. In the present note we give results derived from considerations similar to those in [7]. The underlying concept in the author's papers [5-9] is expressed in

LEMMA 1.1   Consider an infinite region radiating out of the origin in any Euclidean vector space. The set of all lattice point vectors from the origin in that region is precisely the set of positive integer multiples of the visible point vectors (vpv's) in that region.

Immediate *a priori* consequences of this may be written. For example,

$$\sum_{\substack{(a,b)=1 \\ a \geq 0, b \geq 1}} \frac{y^a z^b}{1 - y^a z^b} = \frac{z}{(1-y)(1-z)} , \qquad (1.5)$$

$$\sum_{\substack{(a,b)=1 \\ a \geq 0, b \geq 1 \\ a \geq b}} \frac{y^a z^b}{1 - y^a z^b} = \sum_{k=1}^{\infty} \frac{1 - y^k}{1 - y} z^k = \frac{z}{(1-y)(1-z)} - \frac{yz}{(1-y)(1-yz)} , \qquad (1.6)$$

valid for $|y|$ and $|z| < 1$, $|y|$ and $|yz| < 1$,   respectively

It is straightforward to generalize these results to any number of variables. For example,

$$\sum_{\substack{(a,b,c,d)=1 \\ a,b,c \geq 0, d \geq 1}} \frac{w^a x^b y^c z^d}{1 - w^a x^b y^c z^d} = \frac{z}{(1-w)(1-x)(1-y)(1-z)} , \qquad (1.7)$$

$$\sum_{\substack{(a\,b\,c\,d)=1 \\ a\,b\,c\,0\,d\geq 1 \\ a\,b\,c\,.d}} \frac{w^a x^b y^c z^d}{1 - w^a x^b y^c z^d} = \sum_{k=1}^{\infty} \frac{(1-w^k)(1-x^k)(1-y^k)}{(1-w)(1-x)(1-y)} z^k \,, \tag{1 8}$$

with (1 7) valid for $|w|, |x|, |y|$ and $|z| < 1$ whilst (1 8) is true if absolute values of z, wz, xz, yz, wxz, xyz, wyz, $wxyz$ are all less than unity

Another consequence of the above considerations is

THEOREM 1 2.   Under the same conditions as (1 5) and (1 6) respectively,

$$\prod_{\substack{(a,b,c)=1 \\ a\,b\geq 0, c\geq 1}} (1 - x^a y^b z^c)^{1/c} = (1-z)^{1/(1-x)(1-y)} \,, \tag{1 9}$$

$$\prod_{\substack{(a\,b,c)=1 \\ a,b\geq 0, c\geq 1 \\ a,b<c}} (1 - x^a y^b z^c)^{1/c} = \left(\frac{(1-xz)(1-yz)}{(1-z)(1-xyz)} \right)^{1/(1-x)(1-y)} \tag{1 10}$$

This is true since logarithmic derivatives of both sides of (1 9) and (1.10) yield (1 5) and (1 6) respectively   This establishes the theorem to within a constant factor in each product, and this constant may be determined from allowing both x and y to approach unity

2.   VPV IDENTITIES AND BOUNDARY LIMIT CASES.

The identities of section 1 have interesting cases in which variables other than z approach unity (often the boundary of convergence) given certain modifications   In Campbell [5] we find the identity

$$\prod_{k=1}^{\infty} \left(1 - y^k\right)^{\varphi(k)/k} = \exp\left(\frac{-y}{1-y} \right), \tag{2 1}$$

where φ is the Euler totient function. An equivalent result was stated in Borwein and Borwein [4],

$$\prod_{k=1}^{\infty} \left(1 - y^k\right)^{\mu(k)/k} = e^{-y} \,, \tag{2 2}$$

where μ is the Mobius function. We now establish

THEOREM 2.1

$$\prod_{k=3}^{\infty} \left(1 - z^k\right)^{\varphi_3(k)/k} = \frac{1}{1-z} \exp\left(\frac{z^3}{2(1-z)^2} - \tfrac{1}{2} z - \frac{1}{2z(1-z)} \right), \qquad |z| < 1, \tag{2 3}$$

where $\varphi_3(k)$ denotes for fixed k the number of positive integer solutions of $(a,b,k) = 1$ where $a < b < k$

The function $\varphi_3(k)$ is a natural generalization of the Euler totient function φ which is the number of positive integers less than and relatively prime to k

In the recent paper by the author [7], it was shown that for absolute values of z, yz, xyz, all less than unity,

$$\prod_{\substack{(a,b,c)=1 \\ a\,b<c \\ a\,0\,b\,1\,c}} (1 - x^a y^b z^c)^{-1/c}$$

$$= \frac{(1-yz)^{\frac{1}{(1-x)(1-xy)}} (1-xyz)^{\frac{1}{(1-x)(1-xy)}}}{(1-z)^{\frac{xy^2}{(1-x)(1-xy)}}} \exp\left(\frac{-xyz(2xy - \tfrac{1}{2} yz - \tfrac{1}{2} xyz - z)}{(1-x)(1-y)(1-xy)} \right) \tag{2 4}$$

If we arrange suitably for $x \to 1$, $y \to 1$, in this we should expect to obtain (2 3)   There is no difficulty with the left side, but the right side seems to present formal (if not conceptual) barriers   A way around this is to construct (2 3) by a method offering a geometrical viewpoint   For suitable conditions the logarithm of the left side of (2 3) equals

$$\sum_{k=3}^{\infty} \frac{\varphi_3(k)}{k} \log(1 - z^k)$$

$$= \sum_{k=1}^{\infty} \sum_{\substack{(a,b,c)=1 \\ a \quad b \quad c \\ a \quad 0 \quad b \quad 1 \quad c \quad 2}} 1^{ak} 1^{bk} z^{kc} (kc)^{-1}$$

$$= \sum_{\substack{a \quad b \quad c \\ a \quad 0 \quad b \quad 1 \quad c \quad 2}} 1^{a} 1^{b} z^{c} c^{-1}$$

$$= \frac{z^3}{3} (1^2 1^1) + \frac{z^4}{4} \left(\begin{array}{c} 1^3 1^2 + 1^3 1^1 \\ + 1^2 1^1 \end{array} \right) + \frac{z^5}{5} \left(\begin{array}{c} 1^4 1^1 + 1^4 1^2 + 1^4 1^1 \\ + 1^3 1^2 + 1^3 1^1 \\ + 1^2 1^1 \end{array} \right) + \ldots$$

$$= \frac{z^3}{3} .1 + \frac{z^4}{4} .3 + \frac{z^5}{5} .6 + \frac{z^6}{6} .10 + \frac{z^7}{7} .15 + \ldots$$

$$= \log\left(\frac{1}{1-z} \right) - \frac{1}{2z(1-z)} - \tfrac{1}{2} z + \frac{z^3}{2(1-z)^2} .$$

From here, exponentiation gives the right side of (2 3) which proves the theorem

Futher examples of the vpv identities reduce to new results   An infinite product example given in Campbell [7] is for $|x|$ and $|y| < 1$,

$$\prod_{\substack{(a\ b)=1 \\ a\ b\ Z}} \left(\frac{1}{1 - x^a y^b} \right)^{\frac{1}{a^m b^n}} = \exp\left\{ \left(\sum_{j=1}^{\infty} \frac{x^j}{j^m} \right) \left(\sum_{k=1}^{\infty} \frac{y^k}{k^n} \right) \right\} \quad \text{where} \quad m + n = 1. \tag{2 5}$$

Suppose now we select $x = y$, and $m = n = \tfrac{1}{2}$ Then we have the result valid for $|x| < 1$

$$(1 - x^{1+1})^{-\sqrt{(1)(1)}} (1 - x^{1+2})^{-2\sqrt{(1)(2)}} (1 - x^{1+3})^{-2\sqrt{(1)(3)}} (1 - x^{2+3})^{-2\sqrt{(2)(3)}} (1 - x^{1+4})^{-2\sqrt{(1)(4)}} (1 - x^{3+4})^{-2\sqrt{(3)(1)}} \ldots$$

$$= \exp\left\{ \left(\frac{x}{\sqrt{1}} + \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{3}} + \ldots \right)^2 \right\} \tag{2 6}$$

$$= (1 - x^2)^{a_2} (1 - x^3)^{a_3} (1 - x^4)^{a_4} (1 - x^5)^{a_5} \cdots .$$

where $a_2 = -\sqrt{(1)(1)}$, $a_3 = -2\sqrt{(1)(2)}$, $a_4 = -2\sqrt{(1)(3)}$, $a_5 = -2(\sqrt{(1)(4)} + \sqrt{(2)(3)})$,   ... in which the factors in the products under each square root are coprime and sum to the suffix i of a_i

3.   A DIRICHLET SERIES VERSION OF A GENERAL VPV THEOREM.

In the author's paper [7] we find the fundamental identity in vpv theory,

THEOREM 3 1   If $i = 1, 2, 3, \ldots, n$ then for each $|x_i| < 1$, and $b_i \in C$, with $\sum_{i=1}^{n} b_i = 1$,

$$\prod_{\substack{(a_1, a_2, \ldots, a_n)=1 \\ a_i\ Z}} \left(\frac{1}{1 - x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}} \right)^{\frac{1}{a_1^{b_1} a_2^{b_2} \cdots a_n^{b_n}}} = \exp\left\{ \prod_{i=1}^{n} \left(\sum_{k=1}^{\infty} \frac{x_i^j}{j^{b_i}} \right) \right\}. \tag{3 1}$$

This result came from lemma 1 1 much in the manner shown above for theorem 1 2   A similar line of reasoning applies also to Dirichlet summations over the vpv lattices   We now state

THEOREM 3 2   If $i = 1, 2, \ldots, n$ then for each $b_i \in C$, $\Re b_i > 1$,

$$\sum_{\substack{(a_1, a_2, \ldots, a_n)=1 \\ a_i \in Z^*}} \frac{1}{a_1^{b_1} a_2^{b_2} \cdots a_n^{b_n}} = \frac{\zeta(b_1)\zeta(b_2)\cdots\zeta(b_n)}{\zeta(b_1 + b_2 + \cdots + b_n)}. \tag{3 2}$$

This result appears also to be fundamental in some sense, but is not seen in any of the literature, except for the case with $n = 2$ given recently in Campbell [6]   The latter case is so near the surface it is probably known   The proof of (3 2) follows trivially from applying lemma 1 1 to the left side series after multiplying both sides by

$$\sum_{a \in Z^*} \frac{1}{a^{(b_1 + b_2 + \cdots + b_n)}} = \zeta(b_1 + b_2 + \cdots + b_n). \tag{3 3}$$

It may of course be argued that (3.2) is simply a limiting case of (3.1)   Indeed generalization of the Euler totient function much in the style of our $\varphi_\lambda(k)$ will give the generating Dirichlet series via (3 2)

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