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GENERALIZATION PROPERTIES FOR CERTAIN ANALYTIC FUNCTIONS

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ABSTRACT. The object of the present paper is to give some generalizations of results for certain analytic functions which were proved by Saitoh (Math. Japon. 35 (1990), 1073-1076). Our results contain some corollaries as the special cases.

KEY WORDS AND PHRASES. Analytic function, open unit disk, regular, complex valued function.
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1. INTRODUCTION.

Let $A(n)$ be the class of functions of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \qquad (n \in N = \{1, 2, 3, ...\}) \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$. We introduce the function $F(\alpha, \beta; z)$ defined by

$$F(\alpha, \beta; z) = \alpha f(z) + \beta z f'(z) \tag{1.2}$$

for $f(z) \in A(n)$, where α and β are complex numbers.

For $\beta \in R$ (the set of all real numbers) and $\alpha = 1 - \beta$, Owa [2] has shown some properties for $F(\alpha, \beta; z)$. Recently, for $\alpha = 1 - \beta$ and $\beta \in C$ (the set of all complex numbers), Saitoh [3] has proved

THEOREM A. If $f(z) \in A(n)$ and

$$Re\left\{\frac{F(1-\beta, \beta; z)}{z}\right\} > \alpha \qquad (z \in U) \tag{1.3}$$

for some $\alpha (0 \le \alpha < 1)$ and $\beta \,(Re(\beta) \ge 0)$, then

$$Re\left\{\frac{f(z)}{z}\right\} > \frac{2\alpha + nRe(\beta)}{2 + nRe(\beta)} \qquad (z \in U)\ . \tag{1.4}$$

THEOREM B. If $f(z) \in A(n)$ and

$$Re\left\{\frac{F(1-\beta, \beta; z)}{z}\right\} < \alpha \qquad (z \in U) \tag{1.5}$$

for some $\alpha (\alpha > 1)$ and $\beta (Re(\beta) \ge 0)$, then

$$Re\left\{\frac{f(z)}{z}\right\} < \frac{2\alpha + nRe(\beta)}{2 + nRe(\beta)} \qquad (z \in U) , \tag{1.6}$$

THEOREM C. If $f(z) \in A(n)$ and

$$Re(F'(1-\beta, \beta; z)) > \alpha \qquad (z \in U) \tag{1.7}$$

for some $\alpha (0 \leq \alpha < 1)$ and $\beta (Re(\beta) \geq 0)$, then

$$Re(f'(z)) > \frac{2\alpha + nRe(\beta)}{2 + nRe(\beta)} \qquad (z \in U) . \tag{1.8}$$

THEOREM D. If $f(z) \in A(n)$ and

$$Re(F'(1-\beta, \beta; z)) < \alpha \qquad (z \in U) \tag{1.9}$$

for some $\alpha (\alpha > 1)$ and $\beta (Re(\beta) \geq 0)$, then

$$Re(f'(z)) < \frac{2\alpha + nRe(\beta)}{2 + nRe(\beta)} \qquad (z \in U) . \tag{1.10}$$

In the present paper, we give the generalizations of the above results by Saitoh [3].

2. GENERALIZATION PROPERTIES

To derive our generalizations, we have to recall here the following lemma by Miller and Mocanu [1].

LEMMA I. Let $\phi(u, v)$ be a complex valued function,

$$\phi : D \to C , \qquad D \subset C^2 \qquad (C \text{ is the complex plane}) ,$$

and let $u = u_1 + iu_2$ and $v = v_1 + iv_2$. Suppose that the function $\phi(u, v)$ satisfies

(i) $\phi(u, v)$ is continuous in D;

(ii) $(1, 0) \in D$ and $Re(\phi(1, 0)) > 0$;

(iii) $Re(\phi(iu_2, v_1)) \leq 0$ for all $(iu_2, v_1) \in D$ and such that

$$v_1 \leq -n(1 + u_2^2)/2 .$$

Let $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \ldots$ be regular in U such that $(p(z), zp'(z)) \in D$ for all $z \in U$. If $Re(\phi(p(z), zp'(z))) > 0 \, (z \in U)$, then $Re(p(z)) > 0 \, (z \in U)$.

Now, we derive

THEOREM 1. If $f(z) \in A(n)$, $\alpha \in C$, $\beta \in C \, (Re(\beta) \geq 0)$, $\alpha + \beta \in R$, and

$$Re\left\{\frac{F(\alpha, \beta; z)}{z}\right\} > \gamma \qquad (z \in U) \tag{2.1}$$

for some $\gamma (\gamma < \alpha + \beta)$, then

$$Re\left\{\frac{f(z)}{z}\right\} > \frac{2\gamma + nRe(\beta)}{2(\alpha + \beta) + nRe(\beta)} \qquad (z \in U) . \tag{2.2}$$

PROOF. Define the function $p(z)$ by

$$\frac{f(z)}{z} = \delta + (1 - \delta)p(z) , \tag{2.3}$$

where

$$\delta = \frac{2\gamma + nRe(\beta)}{2(\alpha + \beta) + nRe(\beta)} < 1 . \tag{2.4}$$

Then $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \dots$ is regular in U, and

$$\begin{aligned}\frac{F(\alpha,\beta;z)}{z} &= \alpha\frac{f(z)}{z} + \beta f'(z)\\ &= (\alpha+\beta)\delta + (\alpha+\beta)(1-\delta)p(z) + \beta(1-\delta)zp'(z)\ . \end{aligned} \tag{2.5}$$

Therefore, we have

$$Re\left\{\frac{F(\alpha,\beta;z)}{z} - \gamma\right\} = Re\{(\alpha+\beta)\delta - \gamma + (\alpha+\beta)(1-\delta)p(z) + \beta(1-\delta)zp'(z)\} > 0\ . \tag{2.6}$$

If we define the function $\phi(u, v)$ by

$$\phi(u,v) = (\alpha+\beta)\delta - \gamma + (\alpha+\beta)(1-\delta)u + \beta(1-\delta)v \tag{2.7}$$

with $u = u_1 + iu_2$ and $v = v_1 + iv_2$, then

(i) $\phi(u, v)$ is continuous in $D = C^2$;

(ii) $(1,0) \in D$ and $Re(\phi(1,0)) = (\alpha+\beta) - \gamma > 0$;

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq -n(1+u_2^2)/2$,

$$\begin{aligned} Re(\phi(iu_2,v_1)) &= Re\{(\alpha+\beta)\delta - \gamma + \beta(1-\delta)v_1\}\\ &\leq (\alpha+\beta)\delta - \gamma - (1-\delta)Re(\beta)\frac{n(1+u_2^2)}{2}\\ &= -\frac{(1-\delta)nRe(\beta)}{2}u_2^2\\ &\leq 0\ . \end{aligned}$$

Thus the function $\phi(u, v)$ satisfies the conditions in the lemma. This gives that $Re(p(z)) > 0\ (z \in U)$, so that

$$Re\left\{\frac{f(z)}{z}\right\} > \delta = \frac{2\gamma + nRe(\beta)}{2(\alpha+\beta) + nRe(\beta)} \qquad (z \in U)\ . \tag{2.8}$$

REMARK. Letting $\alpha = 1 - \beta$ in Theorem 1, we have Theorem A due to Saitoh [3].

Taking $\alpha = \overline{\beta}$ in Theorem 1, we have

COROLLARY 1. If $f(z) \in A(n)$, $\beta \in C\ (Re(\beta) > 0)$, and

$$Re\left\{\frac{F(\overline{\beta},\beta;z)}{z}\right\} > \gamma \qquad (z \in U) \tag{2.9}$$

for some $\gamma(\gamma < 2Re(\beta))$, then

$$Re\left\{\frac{f(z)}{z}\right\} > \frac{2 + nRe(\beta)}{(4+n)Re(\beta)} \qquad (z \in U)\ . \tag{2.10}$$

Further, if

$$Re\left\{\frac{F(\overline{\beta},\beta;z)}{z}\right\} > \frac{3}{2}Re(\beta) \qquad (z \in U)\ , \tag{2.11}$$

then

$$Re\left\{\frac{f(z)}{z}\right\} > \frac{3+n}{4+n} \qquad (z \in U)\ . \tag{2.12}$$

Next, we prove

THEOREM 2. If $f(z) \in A(n)$, $\alpha \in C$, $\beta \in C\,(Re(\beta) \geq 0)$, $\alpha + \beta \in R$, and

$$Re\left\{\frac{F(\alpha,\beta;z)}{z}\right\} < \gamma \qquad (z \in U) \tag{2.13}$$

for some $\gamma(\gamma > \alpha + \beta)$, then

$$Re\left\{\frac{f(z)}{z}\right\} < \frac{2\gamma + nRe(\beta)}{2(\alpha+\beta) + nRe(\beta)} \qquad (z \in U)\,. \tag{2.14}$$

PROOF. Let us define the function $p(z)$ by

$$\frac{f(z)}{z} = \delta + (1-\delta)p(z) \tag{2.15}$$

with

$$\delta = \frac{2\gamma + nRe(\beta)}{2(\alpha+\beta) + nRe(\beta)} > 1\,. \tag{2.16}$$

It follows from (2.15) that $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \ldots$ is regular in U, and

$$Re\left\{\gamma - \frac{F(\alpha,\beta;z)}{z}\right\} = Re\{\gamma - (\alpha+\beta)\delta - (\alpha+\beta)(1-\delta)p(z) - \beta(1-\delta)zp'(z)\} > 0\,. \tag{2.17}$$

Define the function $\phi(u,v)$ by

$$\phi(u,v) = \gamma - (\alpha+\beta)\delta - (\alpha+\beta)(1-\delta)u - \beta(1-\delta)v \tag{2.18}$$

with $u = u_1 + iu_2$ and $v = v_1 + iv_2$. Then

(i) $\phi(u,v)$ is continuous in $D = C^2$;

(ii) $(1,0) \in D$ and $Re(\phi(1,0)) = \gamma - (\alpha+\beta) > 0$;

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq -n(1+u_2^2)/2$,

$$\begin{aligned} Re(\phi(iu_2,v_1)) &= Re\{\gamma - (\alpha+\beta)\delta - \beta(1-\delta)v_1\} \\ &\leq \gamma - (\alpha+\beta)\delta + (1-\delta)Re(\beta)\frac{n(1+u_2^2)}{2} \\ &= \frac{(1-\delta)nRe(\beta)}{2}u_2^2 \\ &\leq 0\,. \end{aligned}$$

Consequently, applying the lemma, we have that $Re(p(z)) > 0 (z \in U)$, which implies, that

$$Re\left\{\frac{f(z)}{z}\right\} < \delta = \frac{2\gamma + nRe(\beta)}{2(\alpha+\beta) + nRe(\beta)} \qquad (z \in U)\,. \tag{2.19}$$

REMARK. If we take $\alpha = 1 - \beta$ in Theorem 2, we have Theorem B by Saitoh [3].

COROLLARY 2. If $f(z) \in A(n)$, $\beta \in C\,(Re(\beta) > 0)$, and

$$Re\left\{\frac{F(\overline{\beta},\beta;z)}{z}\right\} < \gamma \qquad (z \in U) \tag{2.20}$$

for some $\gamma(\gamma > 2Re(\beta))$, then

$$Re\left\{\frac{f(z)}{z}\right\} < \frac{2\gamma + nRe(\beta)}{(4+n)Re(\beta)} \qquad (z \in U)\,. \tag{2.21}$$

Further, if

$$Re\left\{\frac{F(\overline{\beta},\beta;z)}{z}\right\} < \frac{5}{2}Re(\beta) \qquad (z \in U)\ , \tag{2.22}$$

then

$$Re\left\{\frac{f(z)}{z}\right\} < \frac{5+n}{4+n} \qquad (z \in U)\ . \tag{2.23}$$

Employing the same manner as in the proofs of Theorems 1 and 2, we have

THEOREM 3. If $f(z) \in A(n)$, $\alpha \in C$, $\beta \in C\,(Re(\beta) \geq 0)$, $\alpha + \beta \in R$, and

$$Re(F'(\alpha,\beta;z)) > \gamma \qquad (z \in U) \tag{2.24}$$

for some $\gamma(\gamma < \alpha + \beta)$, then

$$Re(f'(z)) > \frac{2\gamma + nRe(\beta)}{2(\alpha+\beta) + nRe(\beta)} \qquad (z \in U)\ . \tag{2.25}$$

REMARK. Taking $\alpha = 1 - \beta$ in Theorem 3, we have Theorem C due to Saitoh [3].

COROLLARY 3. If $f(z) \in A(n)$, $\beta \in C\,(Re(\beta) > 0)$, and

$$Re(F'(\overline{\beta},\beta;z)) > \gamma \qquad (z \in U) \tag{2.26}$$

for some $\gamma(\gamma < 2Re(\beta))$, then

$$Re(f'(z)) > \frac{2\gamma + nRe(\beta)}{(4+n)Re(\beta)} \qquad (z \in U)\ . \tag{2.27}$$

Further, if

$$Re(F'(\overline{\beta},\beta;z)) > \frac{3}{2}Re(\beta) \qquad (z \in U)\ , \tag{2.28}$$

then

$$Re(f'(z)) > \frac{3+n}{4+n} \qquad (z \in U)\ . \tag{2.29}$$

THEOREM 4. If $f(z) \in A(n)$, $\alpha \in C$, $\beta \in C\,(Re(\beta) \geq 0)$, $\alpha + \beta \in R$, and

$$Re(F'(\alpha,\beta;z)) < \gamma \qquad (z \in U) \tag{2.30}$$

for some $\gamma(\gamma > \alpha + \beta)$, then

$$Re(f'(z)) < \frac{2\gamma + nRe(\beta)}{2(\alpha+\beta) + nRe(\beta)} \qquad (z \in U)\ . \tag{2.31}$$

REMARK. Making $\alpha = 1 - \beta$ in Theorem 4, we have Theorem D by Saitoh [3].

COROLLARY 4. If $f(z) \in A(n)$, $\beta \in C\,(Re(\beta) > 0)$, and

$$Re(F'(\overline{\beta},\beta;z)) < \gamma \qquad (z \in U) \tag{2.32}$$

for some $\gamma(\gamma > 2Re(\beta))$, then

$$Re(f'(z)) < \frac{2\gamma + nRe(\beta)}{(4+n)Re(\beta)} \qquad (z \in U)\ . \tag{2.33}$$

Further, if

$$Re(F'(\overline{\beta},\beta;z)) < \frac{5}{2}Re(\beta) \qquad (z \in U)\ , \tag{2.34}$$

then

$$Re(f'(z)) < \frac{5+n}{4+n} \qquad (z \in U)\ . \tag{2.35}$$

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