STRUCTURE OF THE ANTIEIGENVECTORS OF A STRICTLY ACCRETIVE OPERATOR

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ABSTRACT. A necessary and sufficient condition that a vector f is an antieigenvector of a strictly accretive operator A is obtained. The structure of antieigenvectors of selfadjoint and certain class of normal operators is also found in terms of eigenvectors. The Kantorovich inequality for selfadjoint operators and the Davis's inequality for normal operators are then easily deduced. A sort of uniqueness is also established for the values of Re(Af,f) and ||Af|| if the first antieigenvalue, which is equal to min Re(Af,f)/(||Af||||f||) is attained at the unit vector f.

KEY WORDS AND PHRASES. Antieigenvectors, Accretive Operator.

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1. INTRODUCTION

Let H be a complex Hilbert space with inner product $(,)$, A a strictly accretive operator on H and A^* its adjoint. Let $\phi_A(f) = (Af,f)/(||Af||||f||)$, $f \neq 0$ and $\mu_1(A) = \min \{ \text{Re } \phi_A(f) : f \in H, f \neq 0\}$.

Krein [1] and Gustafson [2] have studied $\mu_1(A)$ and indicated how the knowledge of $\mu_1(A)$ can be useful in the study of the spectrum of certain integral operators, initial value problems and some other areas. Geometrically $\mu_1(A)$ represents the (real) cosine of the largest (real) angle through which an arbitary non-zero vector f can be rotated by action of A. Gustafson calls it the first antieigenvalue of A and f the corresponding antieigenvector (See Remark 4.3, however, at the end of Section 4). The exact value of $\mu_1(A)$ can be found in the case of a selfadjoint A, using the Kantorovich inequality [2,3,4].

For a class of normal operators, Davis [3] obtained an inequality from which $\mu_1(A)$ can be found. He made use of the concept of a shell of a Hilbert space operator [5,6] to arrive at the results. Mirman [7] gave a method of estimation of $\mu_k(A)$, the higher antieigenvalues of A, which is defined by Gustafson as follows :

$$\mu_n(A) = \min \{ \text{Re}(Af,f)/(||Af||||f||) : f \neq 0, f \perp f^{(1)}, f^{(2)} \ldots ,f^{(n-1)}\},$$

where $f^{(k)}$ is the kth antieigenvector.

In the present note we put $\mu_1(A)$ in a greater and clearer perspective by studying the

structure of the vectors that make $Re(Af,f)/(\|Af\|\|f\|)$, $f \neq 0$, stationary. (For definition see Section 2). We call them stationary vectors. Clearly every antieigenvector is a stationary vector. We show that for selfadjoint and normal operators a suitable combination of two distinct eigenvectors yields a stationary vector while the converse is true for all selfadjoint and a certain class of normal operators. Kantorovich inequality and Davis's inequality all follow as immediate consequences. If the first antieigenvalue, namely min Re $\phi_A(f)$ = min $Re(Af,f)/(\|Af\|\|f\|)$, $f \neq 0$ is attained for a unit vector f then it is shown that the values of $Re(Af,f)$ and $\|Af\|$ are unique. An equation characterizing vectors for which $|\phi_A(f)|$ is stationary, is also given. A recursive variational scheme is used to define higher antieigenvalues.

2. STATIONARY VALUES OF Re $\phi_A(f)$

let A be a strictly accretive operator on H and Let $B = (A + A^*)/2$ and $C = A^*A$. Both the operators B and C are obviously positive definite. Re $\phi_A(f)$ represents the cosine of the real angle of rotation of the vector f under the action of A. Re $\phi_A(f)$ is said to have a stationary value at a vector f ($\neq 0$) if the function $w_g(t)$ of a real variable t, defined by

$$w_g(t) = \frac{(Bf + tBg, f + tg)^2}{(Cf + tCg, f + tg)(f + tg, f + tg)} \qquad (2.1)$$

has a stationary value at $t = 0$ for any arbitrary but fixed vector g. In other words we must have $w'_g(0) = 0$ for all g. For $\|f\| = 1$, set $(Bf,f) = b$ and $(Cf,f) = c^2$. With these notations, we see that Re $\phi_A(f)$ is stationary at f iff

$$2c^2 Re(Bf,g) - b\ Re(Cf,g) - bc^2\ Re(f,g) = 0. \qquad (2.2)$$

Since g is arbitrary, we have the following theorem.

THEOREM 2.1. Let $\phi_A(f)$, b and c^2 be defined as above. A unit vector f is a stationary vector of Re $\phi_A(f)$ iff

$$c^2(A + A^*)f - bA^*Af - bc^2 f = 0 \qquad (2.3)$$

Equation (2.3) obviously characterizes the vectors for which Re $\phi_A(f)$ is stationary, in particular, a minimum or a maximum.

COROLLARY 2.1. If for a stationary vector f, $Af = A^*f$ then f is a linear combination of two eigenvectors $Af - c^2/b\ f \pm c/b\ \sqrt{c^2-b^2}\ f$ of A^* with corresponding eigenvalues $c^2/b \pm c/b\ \sqrt{c^2-b^2}$.

If in addition A is normal then f is a linear combination of two eigenvectors $Af - c^2/b\ f \pm c/b\ \sqrt{c^2-b^2}f$ of A with corresponding eigenvalues $c^2b \pm c/b\ \sqrt{c^2-b^2}$.

3. STATIONARY VALUES OF $\Phi_A(f)$ FOR SELFADJOINT OPERATORS

For a selfadjoint operator A we can obtain the structure of the stationary vectors of $Re\phi_A(f)$ (which is obviously equal to $\phi_A(f)$ in this case), in particular, of the antieigenvectors in terms of the eigenvectors. It will follow that $\mu_1(A) = 2\ \sqrt{mM}\ (m + M)^{-1}$, where as usual m, M are the least and the greatest eigenvalues of A respectively. This is the Kantorovich inequality (see [3] and [4]). We now state

THEOREM 3.1. A unit vector f is a stationary vector of a selfadjoint operator A iff there exists two eigenvectors whose appropriate linear combination (in the sense given below) yields f.

PROOF. If f is a stationary vector, in particular an antieigenvector then it satisfies (2.3).

As A is selfadjoint (2.3) reduces to

$\|Af\|^2 2Af - (Af, f)A^2f - (Af.f) \|Af\|^2f = O$ which implies $A^2f - \|Af\|/\phi_A(f)$ $Af \pm \|h\|/\phi_A(f)$ Af $= \|Af\|/\phi_A(f)$ $Af - \|Af\|^2f \pm \|h\|/ \phi_A(f)$ Af where $h = Af - \lambda f$, with $(f,h) = 0$ and $\lambda = (Af,f)$.

that is $A[Af-\|Af\|/\phi_A(f)$ $f \pm \|h\|/\phi_A(f)$ $f] = (\|Af\|\pm \|h\|)/\phi_A(f)$ $[Af- \|Af\|/\phi_A(f) f\pm \|h\|/\phi_A(f) f]$

Let $\mu_1 = (\|Af\| + \|h\|) /\phi_A (f)$, $\mu_2 = (\|Af\| - \|h\|)/\phi_A(f)$

$g_1 = Af - \mu_2 f$, $g_2 = Af - \mu_1 f$, then $Ag_1 = \mu_1 g_1$, $Ag_2 = \mu_2 g_2$ and $f = 1/(\mu_1-\mu_2)$ (g_1-g_2). So that f is a linear combination of two eigenvectors.

Conversely, let $f = \alpha_i e_i + \alpha_j e_j$, with $|\alpha_i|^2 = \dfrac{\lambda_j}{\lambda_i + \lambda_j}$ and $|\alpha_j|^2 = \dfrac{\lambda_i}{\lambda_i + \lambda_j}$.

where e_i and e_j are any two eigenvectors of A corresponding to eigenvalues λ_i and λ_j.

So $b = (Af,f) = \lambda_i|\alpha_i|^2 + \lambda_j|\alpha_j|^2 = \dfrac{2\lambda_i\lambda_j}{\lambda_i + \lambda_j}$ and $c^2 = \lambda_i^2 |\alpha_i|^2 + \lambda_j^2 |\alpha_j|^2$. With these values

we can (2.3) see that the equation (2.3) is satisfied by f. This completes the proof.

The Kantorovich inequality (see [2], [3] and [4]) now follows as a corollary.

COROLLARY 3.1. If A is selfadjoint and $\mu_1(A) = \min\{(Af,f)/(\|Af\|\|f\|) : f \in H, f \neq 0\}$ then $\mu_1(A) = 2\sqrt{mM} / (m + M)$.

PROOF. One need only to observe that $\mu_1(A)$ is stationary value of $\phi_A(f)$ and $2\sqrt{\lambda_i\lambda_j} / (\lambda_i + \lambda_j) = 2 / (\sqrt{\lambda_i} / \sqrt{\lambda_j} + \sqrt{\lambda_j} / \sqrt{\lambda_i})$ assumes the least value for greatest λ_i/λ_j, which in this case is M/m.

4.STATIONARY VALUES OF Re $\phi_A(f)$ FOR A NORMAL OPERATOR

Before we discuss the structure of the stationary vectors in the normal case, we give a few examples to show that the situation in this case can be very different. In these examples e_k' s are eigenvectors of A corresponding to eigenvalues λ_k's, the values of which are clear from the context.

EXAMPLE 4.1. Let A be normal with $Af = \Sigma(1+ik)$ $(f,e_k)e_k$. $i = \sqrt{-1}$, $\|f\| = 1$. So $b = \Sigma |(f,e_k)|^2 = 1$, $c^2 = \Sigma(1+k^2)$ $|(f,ek)|^2$. Hence $b^2/c^2 = 1/[\Sigma(1+k^2)|(f,e_k)|^2]$. Obviously, Re $\phi_A(f)$ attains a minimum at the eigenvector e_n and $\mu_1(A) = 1/(1 + n^2)$. where $n = \max \{k\}$.

EXAMPLE 4.2. Let $Af = (1 + i)$ $(f,e_1)e_1 + (1 - i)$ $(f,e_2)e_2 + (2 + i)$ $(f, e_3)e_3 + (2 - i)$ $(f,e_4)e_4$, where $i = \sqrt{-1}$, $\|f\| = 1$. So $b = 1 + |(f,e_3)|^2 + |(f,e_4)|^2$ and $c^2 = 2 + 3 |(f,e_3)|^2 + 3|(f,e_4)|^2$. Hence b^2/c^2 will be a minimum for $|(f,e_3)| = |(f,e_4)| = 0$, and then $\mu_1(A)=1/\sqrt{2}$. Note that in this case $b = \text{Re } \lambda_i$ and $c^2 = |\lambda_i|^2$.

EXAMPLE 4.3. This is the most important example in this section. It shows that unlike the selfadjoint case, a linear combination of more than two eigenvectors corresponding to dinstinct eigenvalues may exist for the attainment of the minimum of Re $\phi_A(f)$. Let us consider the normal operator A such that

$Af = (1 + i)$ $(f,e_1)e_1 +1/2 (1 + i)$ $(f,e_2)e_2 + (2 + i)$ $(f,e_3)e_3$ with $|(f,e_1)|^2 + |(f,e_2)|^2 +|(f,e_3)|^2 =1$, $i = \sqrt{-1}$.

Clearly $b = \text{Re } (Af,f) = |(f,e_1)|^2 + \dfrac{1}{2}|(f,e_2)|^2 + 2 |(f,e_3)|^2 = 1 + |(f,e_3)|^2$

$-\dfrac{1}{2} | (f,e_2)|^2$

and $c^2 = 2 |(f,e_1)|^2 + \dfrac{1}{2} |(f,e_2)|^2 + 5|(f,e_3)|^2$

$= 2 + 3 |(f,e_3)|^2 - \dfrac{3}{2}|(f,e_2)|^2$

Hence $b^2/c^2 = (1 + k)^2/(2 + 3k)$, where $k = |(f,e_3)|^2 - \dfrac{1}{2}|(f,e_2)|^2$. For a minimum or a maximum.

we must have $k = -1$ or $-1/3$. The case $k = -1$ must be ruled out as in that case $|(f,e_2)|^2 > 1$.

For k = -1/3, we have Re $\phi_A(f)$ = 2/3. Let $|(f,e_1)|^2$ = 1/12, $|(f,e_2)|^2$ = 10/12 and $|(f,e_3)^2$ = $\frac{10}{12}$.

So that unit vector f = $\frac{1}{\sqrt{12}}$ e_1+ $\frac{\sqrt{10}}{\sqrt{12}}$ e_2 + $\frac{1}{\sqrt{12}}$ e_3 will be the first antieigenvector of A. However it is possible to have a combination of only two eigenvectors corresponding to two eigenvalues for which the minimum in question is attained. Set $|(f,e_3)|$ = 0, $|(f,e_2)|^2$ = $\frac{2}{3}$ and $|(f,e_1)|^2$ = $\frac{1}{3}$. Clearly f = $\frac{1}{\sqrt{3}}$ e_1+ $\frac{\sqrt{2}}{\sqrt{3}}$ e_2 will be a required vector.

THEOREM 4.1 : Let A be normal with a complete orthonormal set of eigenvectors e_i and corresponding eigenvalues λ_i such that for any unit vector f ∈ H, Af = Σ λ_i $(f,e_i)e_i$.

If Re $\phi_A(f)$ is stationary at f, and f is not an eigenvector of A, then either f is a linear combination of two eigenvectors corresponding to two conjugate eigenvalues or there exists a linear combination of two eigenvectors corresponding to two eigenvalues with unequal real parts and moduli, say g, such that Re $\phi_A(g)$ = Re $\phi_A(f)$ and Re ϕ_A is stationary at g.

Further, we have the relation

$$Re^2\phi_A (f) = \frac{4\lambda\mu \ (\mu\cos\theta - \lambda\cos\phi) \ (\mu \ \cos\phi - \lambda\cos\theta)}{(\lambda^2 - \mu^2)^2}$$

if $\lambda e^{i\theta}$ and $\mu e^{i\phi}$ are the distinct eigenvalues referred to above and are such that $\lambda \neq \mu$ and $\lambda \cos\theta \neq \mu \cos\phi$.

PROOF. If Re $\phi_A(f)$ is stationary at f, we must have

$$c^2(A+A^*)f - bA^*Af - bc^2f = 0 \qquad (4.1)$$

Substitute f = Σ $(f,e_i)e_i$, Af = Σ λ_i $(f,e_i)e_i$ and A*f = Σ $\bar{\lambda}_i$ $(f,e_i)e_i$, in (4.1) to get

$$2c^2 \ Re \ \lambda_i - b \ |\lambda_i|^2 - bc^2 = 0 \qquad (4.2)$$

if $(f,e_i) \neq 0$

As in example 4.1, (4.2) may be satisfied if Af = λf, b = Re λ, c^2 = $|\lambda|^2$. If f is not an eigenvector then (4.2) may be satisfied by only one pair (Re λ_i,$|\lambda_i|^2$) in which case, since f is not an eigenvector, $(f,e_i) \neq 0$ for at least two e_i's corresponding to two distinct λ_i's. So, the only way this can happen is that Af = $\lambda(f,e_1)e_1$ + $\bar{\lambda}(f,e_2)e_2$ where the renumbering of e's is obvious. Here b = Re λ, c^2 = $|\lambda|^2$. This is the case in example 4.2.

If (4.2) is satisfied by more than one pair (Re λ_i, $|\lambda_i|^2$) then as in example 4.3, the stationary vector f may be a linear combination of more than two eigenvectors corresponding to distinct eigenvalues. We shall show, however, that there always exist a suitable linear combination g of two eigenvectors corresponding to two eigenvalues of distinct moduli and real parts, such that Re $\phi_A(g)$ = Re $\phi_A(f)$ and Re ϕ_A is stationary at g. Let λ_i = $\lambda e^{i\theta}$ and λ_j = $\mu e^{i\phi}$, $\lambda \neq \mu$ and $\lambda \cos\theta \neq \mu \cos\phi$ satisfy (4.2). We will find α_i and α_j such that g = $\alpha_i e_i$ + $\alpha_j e_j$, $|\alpha_i|^2$ +$|\alpha_j|^2$ = 1, b = Re (Af,g) = Re(Ag,g) and c^2 = (Af,Af) = (Ag,Ag) and (4.1) is satisfied for g. Now b = $|\alpha_i|^2$ $\lambda \cos\theta$ + $|\alpha_j|^2\mu \cos\phi$ and c^2 = $|\alpha_i|^2\lambda^2$ +$|\alpha_j|^2\mu^2$.

Since $|\alpha_i|^2$ + $|\alpha_j|^2$ = 1, we get

$$|\alpha_i|^2 = \frac{b- \mu \cos \phi}{\lambda \cos\theta - \mu \cos \phi} \quad \text{and also} \quad |\alpha_i|^2 = \frac{c^2 - \mu^2}{\lambda^2 - \mu^2} .$$

But since $2c^2\lambda \cos\theta - b\lambda^2 - bc^2 = 0$ and $2c^2\mu \cos\phi - b\mu^2 - bc^2 = 0$, the value of $|\alpha_1|^2$ is the same. Hence a g is obtained. A simple calculation further shows that

$$\frac{b^2}{c^2} = \frac{4\lambda\mu (\mu \cos\theta - \lambda \cos\phi) (\mu \cos\phi - \lambda \cos\theta)}{(\lambda^2 - \mu^2)^2} \tag{4.3}$$

and that completes the proof.

REMARK 4.1. It may be noted that (4.3) is equivalent to the inequality of Davis [3] when b^2/c^2 is a minimum of Re $\phi_A(f)$ without the restrictive conditions (in part (ii) of his Theorem).

REMARK 4.2. The points of the line $2c^2X - bY - bc^2 = 0$ lying in the region $Y \geq X^2$ are such that

$$\frac{c^2 - c^2 (1 - (b/c)^2)^{\frac{1}{2}}}{b} \leq X \leq \frac{c^2 + c^2 (1 - (b/c)^2)^{\frac{1}{2}}}{b}$$

Hence all eigenvalues λ with Reλ lying between the above limits and Imλ determined from (4.2) may contribute corresponding eigenvectors to yield a stationary vector of Re $\phi_A(f)$.

REMARK 4.3. Since min Re $\phi_A(f)$ may be attained even for an eigenvector as in example 4.1, Gustafson's choice to call it an antieigenvalue does not seem to be a happy one.

5. FURTHER PROPERTIES AT min ϕ_A (f) AND VARIATIONAL CHARACTERIZATION

Consideration of the second derivative of $w_g(t)$, as defined in equation (2.1), shows that $w_g''(0) \geq 0$ is a necessary condition for a unit vector f to be the first antieigenvector. A rather lengthy calculation shows that then

$$4c^2\text{Re}^2 (Bf\text{-}bf,g) + 2bc^2(Bg,g) - b^2(Cg,g) - b^2c^2(g,g) \geq 0 \tag{5.1}$$

for arbitrary $g \in H$, and b,c defined as before. The quantities b and c are unique in the sense of the following theorem.

THEROM 5.1. If the min $\phi_A(f)$ is attained for a unit vector f with $b = (Bf,f)$ and $c^2 = (Cf,f)$, then there cannot exist unit vectors g such that $(Bg,g) = kb$ and $(Cg,g) = k^2c^2$, $k \neq 1$.

PROOF. If such a g exists then without loss of generality we can have Re (Bf-bf,g) = 0. Hence we get from (5.1), $2kb^2c^2 - k^2b^2c^2 - b^2c^2 \geq 0$ or $- b^2c^2 (k - 1)^2 \geq 0$ which is absurd if $k \neq 1$.

Higher antieigenvalues may be defined in a recursive variational manner as follows :

$$\mu_1 (A) = \min_{f \in H} \text{Re } \phi_A(f) = \text{Re } \phi_A(f_1)$$

$$\mu_{n+1} (A) = \min_{\substack{f \perp f_1.....f_n \\ f \perp Bf_1.......Bf_n}} \text{Re}\phi_A (f)$$

Simplification in the selfadjoint case is obvious where eigenvectors get eliminated pairwise.

Gustafson [2] defines imaginary and total antieigenvalues by min Im $\phi_A(f)$ and min $|\phi_A(f)|$ respectively. While the equation characterizing imaginary antieigenvectors will be similar to our equation (2.3), that for total antieigenvectors will be slightly different. If f is the unit vector for which min $|\phi_A(f)|$ is attained, then we must have

$$2c^2 (b_x A_x - b_Y A_Y)f - |b|^2Cf - |b|^2c^2f = 0$$

where $A_x = (A + A^*)/2$, $A_Y = (A - A^*)/2i$, $C = A^*A$

$b = (Af,f)$, $b_x = \text{Re } (Af,f)$, $b_Y = \text{Im } (Af,f)$ and $c^2 = (Cf,f)$.

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