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THE GERGONNE POINT GENERALIZED THROUGH CONVEX COORDINATES

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ABSTRACT. The Gergonne point of a triangle is the point at which the three cevians to the points of tangency between the incircle and the sides of the triangle are concurrent. In this paper, we follow Konečný [7] in generalizing the idea of the Gergonne point and find the convex coordinates of the generalized Gergonne point. We relate these convex coordinates to the convex coordinates of several other special points of the triangle. We also give an example of relevant computations.

Keywords and phrases. Convex (barycentric) coordinates, cevian, Gergonne point, incircle, Ceva's theorem.

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1. Introduction. When cevians of particular significance in the general triangle (medians, angle bisectors, etc.) are concurrent, their common point is often called a special point of the triangle. Such points have always held interest for geometers. In the past, we have discovered the convex coordinates [7, 6] of several special points at which cevians from the three vertices are concurrent [3, 2]. We choose the terminology "convex coordinates" rather than the more widely used "barycentric" or "trilinear coordinates" because of their relevance to convex sets [9].

Now, let the circle $C(I)$ be the incircle of $\triangle V_1V_2V_3$ as shown below. The cevians from the vertices to the points of tangency on the opposite sides of the triangle are concurrent at the point G which is known as the Gergonne point of the triangle [5, 8].

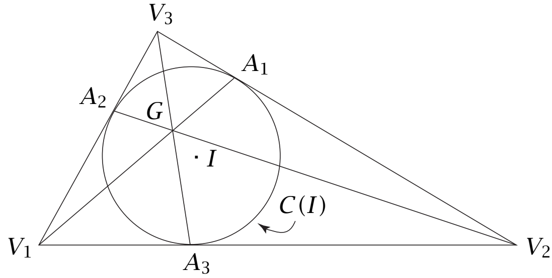

FIGURE 1. Incircle $C(I)$ with Gergonne point G in $\Delta V_1V_2V_3$.

The convex coordinates of a point P in the plane of $\triangle V_1V_2V_3$ and relative to this triangle may be taken as weights which if placed at vertices V_1, V_2, V_3, cause P to become the balance point for the plane. The plane is taken to be horizontal and otherwise weightless. Also, we require that the sum of the weights have unit value. If P belongs to the closed triangular region, then all the three weights are nonnegative.

We denote the weight placed at vertex V_i by α_i. Then point P has convex coordinates $(\alpha_1, \alpha_2, \alpha_3)$ with respect to V_1, V_2, V_3, in that order, and $\alpha_1 + \alpha_2 + \alpha_3 = 1$. For example, the convex coordinates of the vertices V_1, V_2, V_3, are $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, respectively, and the convex coordinates of the centroid of $\triangle V_1V_2V_3$ are $(\alpha_1, \alpha_2, \alpha_3) = (1/3, 1/3, 1/3)$. All points in the exterior of $\triangle V_1V_2V_3$ must have at least one negative coordinate.

Let us return to the Gergonne point G and consider $\triangle V_1V_2V_3$ with its incircle $C(I)$ as redrawn in Figure 2. The point at which $C(I)$ touches the side opposite to V_i is denoted by A_i. The points of tangency divide the sides into segments of lengths x_1, x_2, x_3 as shown in the figure.

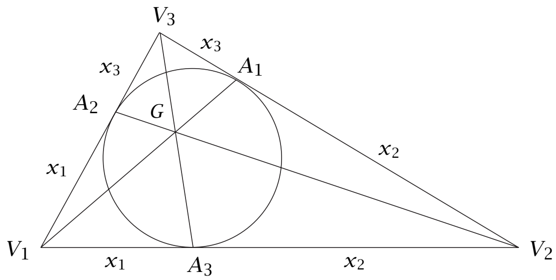

FIGURE 2. $\Delta V_1V_2V_3$, Incircle $C(I)$ and Gergonne point G.

It is not a difficult task to show that the convex coordinates of G have the values

$$\alpha_1 = \frac{x_2x_3}{w}, \qquad \alpha_2 = \frac{x_1x_3}{w}, \qquad \alpha_3 = \frac{x_1x_2}{w}, \tag{1}$$

where $w = x_1x_2 + x_2x_3 + x_1x_3$. [1]

The lengths of the sides and measures of the angles of $\triangle V_1V_2V_3$ are more immediately accessible numbers than are x_1, x_2, and x_3. So, let ℓ_i denote the length of the side opposite to vertex V_i and θ_i denote the measure of the angle at V_i. Figure 3 depicts the triangle again and establishes the notation for the work to follow.

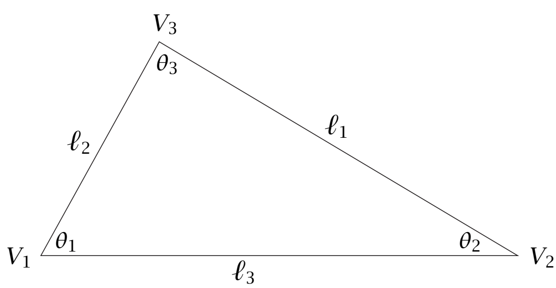

FIGURE 3. $\Delta V_1V_2V_3$ with sides and angles labeled.

The lengths of the sides and the values x_1, x_2, and x_3 are related by the equations

$$\ell_1 = x_2 + x_3, \qquad \ell_2 = x_1 + x_3, \qquad \ell_3 = x_1 + x_2. \tag{2}$$

Thus, $x_1 = (\ell_2+\ell_3-\ell_1)/2$, $x_2 = (\ell_1+\ell_3-\ell_2)/2$, and $x_3 = (\ell_1+\ell_2-\ell_3)/2$. Substitution of these results into the expressions (1) for the convex coordinates of the Gergonne point would not improve their already pleasing appearance.

We have found that convex coordinates provide a straightforward method for investigating special points of triangles.Thus, we read with interest a problem proposed by V. Koneĉný [7] which concerns a generalization of the Gergonne point. In our paper, we find the convex coordinates of Koneĉný's generalized Gergonne point and, in the process, provide an independent proof that the relevant cevians are concurrent.

2. A generalization of the Gergonne point. Let I be the incenter of $\triangle V_1V_2V_3$ and let $D(I)$ be a circle concentric with incircle $C(I)$ as shown in Figure 4. Suppose that lines are drawn through I perpendicular to the sides of the triangle. These lines intersect the sides of $\triangle V_1V_2V_3$ at A_1, A_2, A_3, the points of tangency between the triangle and the incircle and they intersect circle $D(I)$ at points B_1, B_2, and B_3.

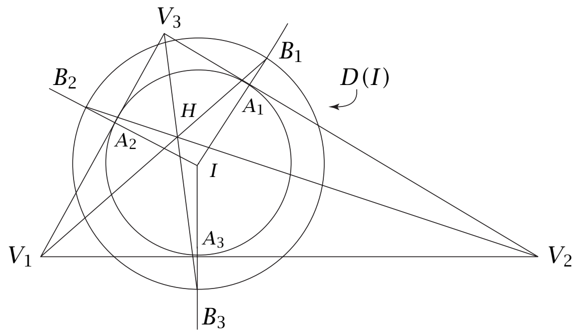

FIGURE 4. $\Delta V_1V_2V_3$ and circle $D(I)$.

Koneĉný's problem is to show that the cevians $\overline{V_1B_1}, \overline{V_2B_2}, \overline{V_3B_3}$ are concurrent. The point H at which the cevians are concurrent is a generalized Gergonne point. We compute the convex coordinates for H and the computational path to our result makes it obvious that the cevians are concurrent.

We begin by noting that there exists $\triangle W_1W_2W_3$ for which $D(I)$ is the incircle and for which H is the Gergonne point. This triangle is similar to $\triangle V_1V_2V_3$; its sides are parallel to the corresponding sides of $\triangle V_1V_2V_3$; and the two triangles are "concentric". If the radius of $C(I)$ is $r > 0$, then the radius of $D(I)$ is $r+t$, where $-r < t$. The corresponding sides of the two triangles are a perpendicular distance of $|t|$ units apart and the similarity ratio for lengths in the two triangles is (length in $\triangle W_1W_2W_3$): (length in $\triangle V_1V_2V_3$)= $(r+t) : r$. The geometry of the two triangles and their incircles is shown in Figure 5.

Points B_1, B_2, B_3 divide the sides of $\triangle W_1W_2W_3$ into segments of lengths y_1, y_2, y_3 just as A_1, A_2, A_3 divide the sides of $\triangle V_1V_2V_3$ into segments of lengths x_1, x_2, x_3. An additional crucial observation is that W_i, V_i, I are collinear for $i = 1, 2, 3$ since rays $\overrightarrow{W_iI}$ and $\overrightarrow{V_iI}$ bisect the congruent vertex angles at W_i and V_i.

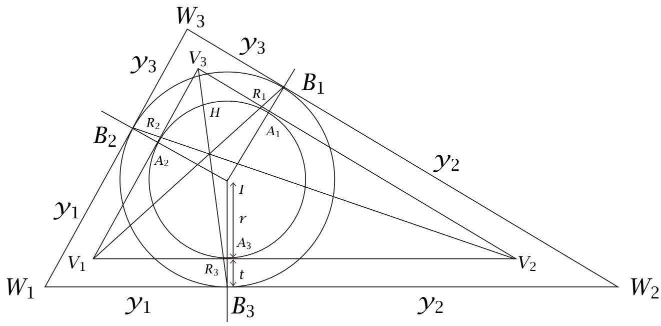

FIGURE 5. The geometry of $\Delta V_1V_2V_3$ and $W_1W_2W_3$.

Let R_i be the point at which $\overline{V_iB_i}$ intersects the side of $\triangle V_1V_2V_3$ opposite to vertex V_i. We must compute the lengths of $\overline{V_1R_3}, \overline{V_2R_3}, \overline{V_2R_1}, \overline{V_3R_1}, \overline{V_3R_2}$, and $\overline{V_1R_2}$. These segments serve as lever arms when weights $\alpha_1, \alpha_2, \alpha_3$ are placed at V_1, V_2, V_3 and the cevians $\overline{V_1R_1}, \overline{V_2R_2}, \overline{V_3R_3}$ are taken to define balance lines. We show the calculations for the length of $\overline{V_1R_3}$ in some detail and then simply state the other five lengths.

We extend $\overline{V_1V_3}$ to intersect $\overline{W_1W_2}$ at point P and we draw angle bisector $\overline{W_1I}$ through V_1 as shown in Figure 6.

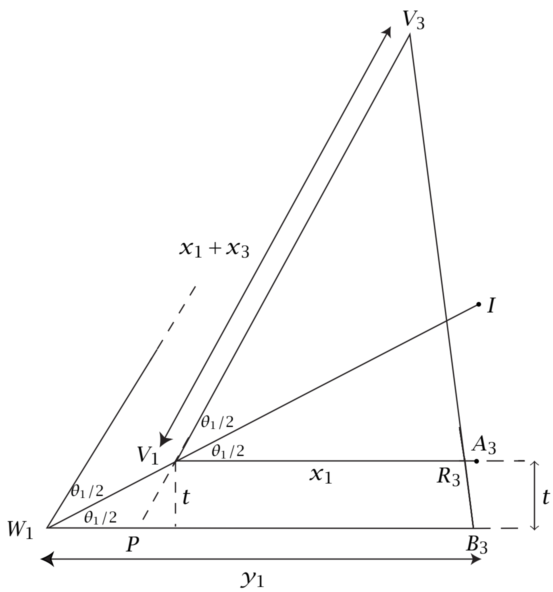

FIGURE 6. The geometry for computing V_1R_3.

Since triangles $W_1W_2W_3$ and $V_1V_2V_3$ are similar with similarity ratio $(r+t):r$, it should be clear that

$$y_1 = \left(\frac{r+t}{r}\right) V_1A_3 = \left(\frac{r+t}{r}\right) x_1, \tag{3}$$

where the meaning of x_i for $i = 1,2,3$ is given in Figure 2. The length of $\overline{PB_3}$ is given by

$$PB_3 = y_1 - t\left(\cot\frac{\theta_1}{2} - \cot\theta_1\right), \tag{4}$$

and the length of $\overline{PV_1}$ by $PV_1 = t(\cot(\theta_1/2) - \cot\theta_1)$.

Before proceeding, let us simplify our notation by letting

$$m_1 = \left(\cot\frac{\theta_1}{2} - \cot\theta_1\right). \tag{5}$$

Then $PB_3 = y_1 - tm_1$ and $PV_1 = tm_1$.

Triangles PV_3B_3 and $V_1V_3R_3$ are also similar. Therefore,

$$V_1R_3 = \frac{(x_1+x_3)PB_3}{x_1+x_3+tm_1}, \tag{6}$$

where $x_1 + x_3 = V_1V_3 = \ell_2$.

Appropriate substitutions yield

$$V_1R_3 = \frac{(x_1+x_3)[((r+t)/r)x_1 - tm_1]}{x_1+x_3+tm_1}, \tag{7}$$

so that all values except t depend only upon the geometry of $\triangle V_1V_2V_3$.

Letting $m_i = \cot(\theta_i/2) - \cot\theta_i$ for $i = 1,2,3$, we give the results of the other computations for lever arms

$$\begin{aligned}
V_2R_3 &= \frac{(x_2+x_3)[((r+t)/r)x_2 - tm_2]}{x_2+x_3+tm_2}, \\
V_2R_1 &= \frac{(x_1+x_2)[((r+t)/r)x_2 - tm_2]}{x_1+x_2+tm_2}, \\
V_3R_1 &= \frac{(x_1+x_3)[((r+t)/r)x_3 - tm_3]}{x_1+x_3+tm_3}, \\
V_3R_2 &= \frac{(x_2+x_3)[((r+t)/r)x_3 - tm_3]}{x_2+x_3+tm_3}, \\
V_1R_2 &= \frac{(x_1+x_2)[((r+t)/r)x_1 - tm_1]}{x_1+x_2+tm_1}.
\end{aligned} \tag{8}$$

At first glance, these expressions seem quite complicated but an examination of the subscripts should reveal their symmetry. The appearance of such symmetry gives confidence in the computations thus far. However, for the next calculations, a few changes in form are helpful. We make the substitutions suggested by equations (1) and (9). The result, denoted by (9), follows from the triangles shown in Figure 7.

We have extended $\overline{V_3V_2}$ to B on $\overline{W_1W_2}$ and drawn $\overline{AV_2}$ parallel to $\overline{V_1V_3}$. Thus, $\triangle V_1V_2V_3 \sim \triangle ABV_2$. The length of $\overline{V_2A}$ is tm_1 and the length of $\overline{V_2B}$ is tm_2. From the similar triangles, we have $t\ell_2m_2 = t\ell_1m_1$. Taking the smaller triangle at a different vertex of $\triangle V_1V_2V_3$ yields

$$t\ell_1m_1 = t\ell_2m_2 = t\ell_3m_3 = k. \tag{9}$$

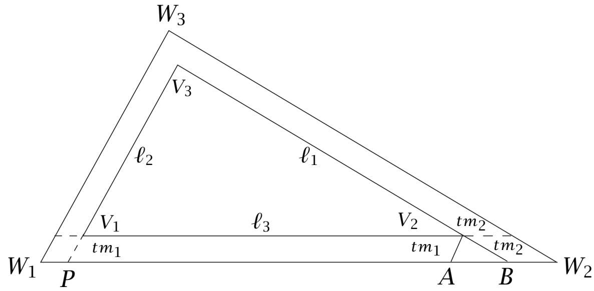

FIGURE 7. Similar triangles $V_1V_2V_3$ and ABV_2.

Equations (7) and (8) become

$$\begin{aligned}
V_1R_3 &= \frac{\ell_2[\ell_1((r+t)/r)x_1-k]}{\ell_1\ell_2+k},\\
V_2R_3 &= \frac{\ell_1[\ell_2((r+t)/r)x_2-k]}{\ell_1\ell_2+k},\\
V_2R_1 &= \frac{\ell_3[\ell_2((r+t)/r)x_2-k]}{\ell_2\ell_3+k},\\
V_3R_1 &= \frac{\ell_2[\ell_3((r+t)/r)x_3-k]}{\ell_2\ell_3+k},\\
V_3R_2 &= \frac{\ell_1[\ell_3((r+t)/r)x_3-k]}{\ell_1\ell_3+k},\\
V_1R_2 &= \frac{\ell_3[\ell_1((r+t)/r)x_1-k]}{\ell_1\ell_3+k}.
\end{aligned}\tag{10}$$

It becomes clear at this stage that $\overline{V_1R_1}, \overline{V_2R_2}$, and $\overline{V_3R_3}$ must be concurrent. That conclusion follows from Ceva's theorem since

$$\frac{V_1R_3}{V_2R_3}\cdot\frac{V_2R_1}{V_3R_1}\cdot\frac{V_3R_2}{V_1R_2}=1.\tag{11}$$

To find the convex coordinates of H, the generalized Gergonne point at which the cevians are concurrent, we place weights $\alpha_1', \alpha_2', \alpha_3'$ at V_1, V_2, V_3 respectively and require that the cevians $\overline{V_3R_3}$ and $\overline{V_1R_1}$ define balance lines. This means that

$$(V_1R_3)\alpha_1' = (V_2R_3)\alpha_2'\tag{12}$$

and

$$(V_2R_1)\alpha_2' = (V_3R_1)\alpha_3'.\tag{13}$$

The weights are denoted by α_i' since we will not normalize coordinates until we have convinced ourselves that we have a triple of weights with point H as balance point. Then we write $\alpha_i = \alpha_i'/(\alpha_1'+\alpha_2'+\alpha_3')$.

A bit of algebra suggests that

$$\begin{aligned}
\alpha_1' &= (V_2R_3)(V_3R_1) = \frac{\ell_1[\ell_2((r+t)/r)x_2-k]}{\ell_1\ell_2+k}\cdot\frac{\ell_2[\ell_3((r+t)/r)x_3-k]}{\ell_2\ell_3+k},\\
\alpha_2' &= (V_1R_3)(V_3R_1) = \frac{\ell_2[\ell_1((r+t)/r)x_1-k]}{\ell_1\ell_2+k}\cdot\frac{\ell_2[\ell_3((r+t)/r)x_3-k]}{\ell_2\ell_3+k},\\
\alpha_3' &= (V_2R_1)(V_1R_3) = \frac{\ell_3[\ell_2((r+t)/r)x_2-k]}{\ell_2\ell_3+k}\cdot\frac{\ell_2[\ell_1((r+t)/r)x_1-k]}{\ell_1\ell_2+k}.
\end{aligned}\tag{14}$$

The reader may satisfy himself that $\alpha_1',\alpha_2',\alpha_3'$ do indeed satisfy equations (12) and (13). The two cevians define balance lines and their point of intersection must be the balance point. The third cevian defines a balance line if and only if

$$(V_3R_2)\alpha_3' = (V_1R_2)\alpha_1'. \tag{15}$$

Substitution of the values for $\alpha_1',\alpha_2',\alpha_3'$ into equation (15) yields $(V_3R_2)(V_2R_1)\times(V_1R_3) = (V_1R_2)(V_2R_3)(V_3R_1)$ which holds true if and only if

$$\frac{V_1R_3}{V_2R_3}\cdot\frac{V_2R_1}{V_3R_1}\cdot\frac{V_3R_2}{V_1R_2} = 1. \tag{16}$$

This last equation is valid by Ceva's theorem. Thus, the convex coordinates of the generalized Gergonne point H are given by

$$\alpha_1 = \frac{\alpha_1'}{\alpha_1'+\alpha_2'+\alpha_3'},\qquad \alpha_2 = \frac{\alpha_2'}{\alpha_1'+\alpha_2'+\alpha_3'},\qquad \alpha_3 = \frac{\alpha_3'}{\alpha_1'+\alpha_2'+\alpha_3'}, \tag{17}$$

where the definitions of $\alpha_1',\alpha_2',\alpha_3'$ are given by equation (14).

3. Three checks and an example

Check 1. if $\triangle V_1V_2V_3$ is equilateral, then $x_1 = x_2 = x_3$ and $\ell_1 = \ell_2 = \ell_3$ which implies that $\alpha_1 = \alpha_2 = \alpha_3 = 1/3$ as desired since the symmetry of the triangle forces H to coincide with the centroid.

Check 2. If $t = 0$, H becomes the Gergonne point. Letting $t = 0$ means that $k = 0$. Then equations (7), (8), and (14) imply that $\alpha_1,\alpha_2,\alpha_3$ have the values given by equations (1).

Check 3. If $t \to -r$. Then H approaches the incenter I of $\Delta V_1V_2V_3$ and the convex coordinates $(\alpha_1,\alpha_2,\alpha_3)$ approach the convex coordinates of the incenter,

$$\left(\frac{\ell_1}{\ell_1+\ell_2+\ell_3},\frac{\ell_2}{\ell_1+\ell_2+\ell_3},\frac{\ell_3}{\ell_1+\ell_2+\ell_3}\right)\cdot[4] \tag{18}$$

Example. Let $\triangle V_1V_2V_3$ be the 3-4-5 right triangle shown in Figure 8. Observation and a bit of computation yield that $\ell_1 = 5$, $\ell_2 = 3$, $\ell_3 = 4$, r = radius of incircle $C(I) = 1$, $x_1 = 1$, $x_2 = 3$, $x_3 = 2$, $\theta_1 = 90°$, $\cot\theta_1 = 0$, $\cot(\theta_1/2) = 1$.

Let the radius of $D(I)$ be 2, which implies that $t = 1$. Then $k = t\ell_1 m_1 = t\ell_1(\cot(\theta_1/2) - \cot\theta_1) = 5$.

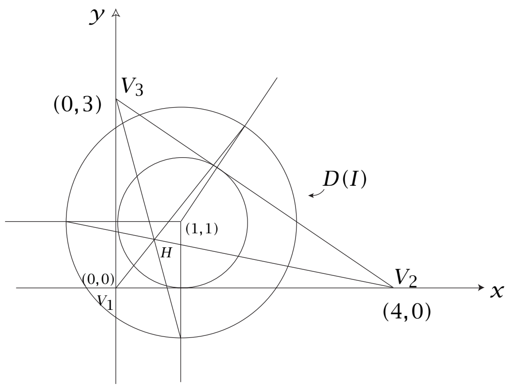

FIGURE 8. The 3-4-5 right triangle located in the Cartesian Plane.

From equations (14) and (15), we obtain the convex coordinates of point H to be $\alpha_1 = 143/228$, $\alpha_2 = 33/228$, $\alpha_3 = 52/228$. Then the Cartesian coordinates of H are

$$\begin{aligned}(x,y) &= \left(0\cdot\alpha_1 + 4\cdot\alpha_2 + 0\cdot\alpha_3, 0\cdot\alpha_1 + 0\cdot\alpha_2 + 3\cdot\alpha_3\right)\\ &= \left(4\cdot\frac{33}{228}, 3\cdot\frac{52}{228}\right) = \left(\frac{11}{9}, \frac{13}{19}\right).\end{aligned} \tag{19}$$

These Cartesian coordinates can also be found by analytic geometry as a further check upon the accuracy of our computations.

REFERENCES

[1] J. N. Boyd, *Convex coordinates of the Gergonne point*, New York State Mathematics Teachers' Journal **35** (1985), 95–98.

[2] J. N. Boyd, J. S. Cooper, and P. N. Raychowdhury, *Euclidean geometry from classical mechanics*, Internat. J. Math. Ed. Sci. Tech. **20** (1989), 383–390.

[3] J. N. Boyd and P. N. Raychowdhury, *Applications of extended convex coordinates*, Math. Gaz. **71** (1987), 209–214.

[4] ______, *Random processes with convex coordinates on triangular graphs*, Internat. J. Math. Math. Sci. **20** (1997), no. 2, 385–392. MR 97k:60187. Zbl 884.60068.

[5] H. S. M. Coxeter and S. L. Greitzer, *Geometry revisited*, vol. XIV, Random House: The L. W. Singer Company, New York, 1967. Zbl 166.16402.

[6] M. Hausner, *The center of mass and affine geometry*, Amer. Math. Monthly **69** (1962), 724–737. Zbl 109.38903.

[7] V. Koneĉny, *Problem department*, Math. Mag. **63** (1990), 130–131.

[8] J. R. Smart, *Modern geometry*, Brook/Cole Publishing Co., Pacific Grove, California, 1994. Zbl 863.51001.

[9] I. M. Yaglom and V. G. Boltyanskii, *Vypuklye figury. [Convex figures.]*, Gosudarstv. Izdat. Tehn. Teor. Lit., Moscow, Leningrad, 1961 (Russian). MR 14,197d.

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