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ANALOGUES OF SOME FUNDAMENTAL THEOREMS OF SUMMABILITY THEORY

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ABSTRACT. In 1911, Steinhaus presented the following theorem: if A is a regular matrix then there exists a sequence of 0's and 1's which is not A-summable. In 1943, R. C. Buck characterized convergent sequences as follows: a sequence x is convergent if and only if there exists a regular matrix A which sums every subsequence of x. In this paper, definitions for "subsequences of a double sequence" and "Pringsheim limit points" of a double sequence are introduced. In addition, multidimensional analogues of Steinhaus' and Buck's theorems are proved.

Keywords and phrases. Subsequences of a double sequence, Pringsheim limit point, P-convergent, P-divergent, RH-regular.

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1. Introduction. In [2, 3, 4, 5, 8], the 4-dimensional matrix transformation $(Ax)_{m,n} = \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}x_{k,l}$ is studied extensively by Robison and Hamilton. Here we define new double sequence spaces and consider the behavior of 4-dimensional matrix transformations on our new spaces. Such a 4-dimensional matrix A is said to be RH-*regular* if it maps every bounded P-convergent sequence (defined below) into a P-convergent sequence with the same P-limit. In [9] Steinhaus proved the following theorem: if A is a regular matrix then there exists a sequence of 0's and 1's which is not A-summable. This implies that A cannot sum every bounded sequence. In this paper, we prove a theorem for double sequences and 4-dimensional RH-regular matrices that is analogous to Steinhaus' theorem. One of the fundamental facts of sequence analysis is that if a sequence is convergent to L, then all of its subsequences are convergent to L. In a similar manner, R. C. Buck [1] characterized convergent sequences by: a sequence x is convergent if and only if there exists a regular matrix A which sums every subsequence of x. We characterize P-convergent double sequences as follows: first, we prove that a double sequence x is P-convergent to L if all of its subsequences are P-convergent to L; then we prove that a double sequence x is P-convergent if there exists an RH-regular matrix A which sums every subsequence of x. In addition, we provide definitions for "subsequences" and "Pringsheim limit points" of double sequences and for divergent double sequence.

2. Definitions, notations, and preliminary results

DEFINITION 2.1 (Pringsheim, 1900). A double sequence $x = [x_{k,l}]$ has Pringsheim limit L (denoted by P-lim $x = L$) provided that given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$|x_{k,l} - L| < \epsilon$ whenever $k, l > N$. We describe such an x more briefly as "P-convergent."

Definition 2.2 (Pringsheim, 1900). A double sequence x is called definite divergent, if for every (arbitrarily large) $G > 0$ there exist two natural numbers n_1 and n_2 such that $|x_{n,k}| > G$ for $n \geq n_1,\ k \geq n_2$.

Definition 2.3. The sequence y is a subsequence of the double sequence x provided that there exist two increasing double index sequences $\{n_j^i\}$ and $\{k_j^i\}$ such that $n_0^1 = k_0^1 = n_{-1}^0 = k_{-1}^0 = 0$ and

n_1^i and k_1^i are both chosen such that $\max\{n_{2i-3}^{i-1}, k_{2i-3}^{i-1}\} < n_1^i, k_1^i$,
n_2^i and k_2^i are both chosen such that $\max\{n_1^i, k_1^i\} < n_2^i, k_2^i$,
n_3^i and k_3^i are both chosen such that $\max\{n_2^i, k_2^i\} < n_3^i, k_3^i$,
$\vdots$
n_{2i-1}^i and k_{2i-1}^i are both chosen such that $\max\{n_{2(i-1)}^i, k_{2(i-1)}^i\} < n_{2i-1}^i, k_{2i-1}^i$, with

$$\begin{aligned}
&y_{1,i} = x_{n_1^i,k_1^i}, \quad y_{2,i} = x_{n_2^i,k_2^i}, \quad y_{3,i} = x_{n_3^i,k_3^i},\\
&\quad \vdots\\
&y_{i,i} = x_{n_i^i,k_i^i}, \quad y_{i,i-1} = x_{n_{i+1}^i,k_{i+1}^i},\\
&\quad \vdots\\
&y_{i,2i-1} = x_{n_{2i-1}^i,k_{2i-1}^i}
\end{aligned}$$

for $i = 1, 2, 3, \ldots$.

A double sequence x is bounded if and only if there exists a positive number M such that $|x_{k,l}| < M$ for all k and l. Define

$$\begin{aligned}
S''\{x\} &= \{\text{all subsequences of } x\};\\
C'' &= \{\text{all bounded P-convergent sequences}\};\\
C''_A &= \left\{x_{k,l} : (Ax)_{m,n} = \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}x_{k,l} \text{ is P-convergent}\right\}.
\end{aligned} \tag{2.1}$$

See Figure 1 for an illustration of the procedure for selecting terms of a subsequence. A 2-dimensional matrix transformation is said to be regular if it maps every convergent sequence into a convergent sequence with the same limit. The Silverman-Toeplitz theorem [6] characterizes the regularity of 2-dimensional matrix transformations. In 1926, Robison presented a 4-dimensional analog of regularity for double sequences in which he added an additional assumption of boundedness. This assumption was made because a double sequence which is P-convergent is not necessarily bounded. The definition of the regularity for 4-dimensional matrices will be stated below, with the Robison-Hamilton characterization of the regularity of 4-dimensional matrices.

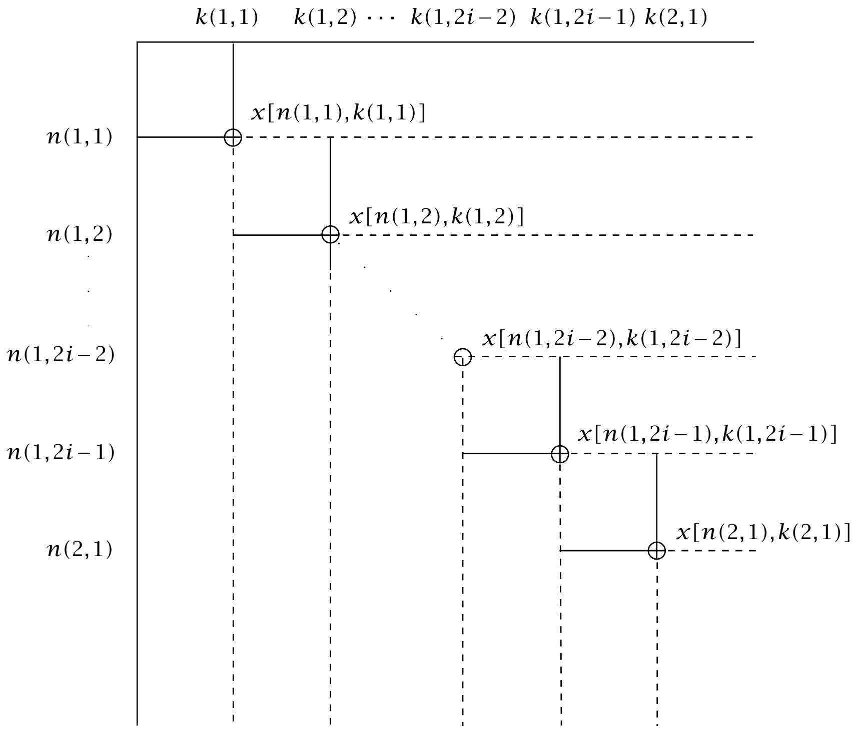

FIGURE 1. The selection process of terms for subsequence y of x, where $x[n(i,j),k(i,j)] = x_{n^i_j,k^i_j}$, $n(i,j) = n^i_j$, $k(i,j) = k^i_j$.

DEFINITION 2.4. The 4-dimensional matrix A is said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.

THEOREM 2.1 (Hamilton [2], Robison [8]). *The 4-dimensional matrix A is* RH*-regular if and only if*

RH$_1$*:* P-$\lim_{m,n} a_{m,n,k,l} = 0$ *for each k and l;*

RH$_2$*:* P-$\lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} = 1$*;*

RH$_3$*:* P-$\lim_{m,n} \sum_{k=0}^{\infty} |a_{m,n,k,l}| = 0$ *for each l;*

RH$_4$*:* P-$\lim_{m,n} \sum_{l=0}^{\infty} |a_{m,n,k,l}| = 0$ *for each k;*

RH$_5$: $\sum_{k,l=0,0}^{\infty,\infty} |a_{m,n,k,l}|$ *is* P*-convergent;*

RH$_6$*: there exist finite positive integers A and B such that $\sum_{k,l>B} |a_{m,n,k,l}| < A$.*

REMARK 2.1. The definition of a Pringsheim limit point can also be stated as follows: β is a Pringsheim limit point of x provided that there exist two increasing index sequences $\{n_i\}$ and $\{k_i\}$ such that $\lim_i x_{n_i,k_i} = \beta$.

DEFINITION 2.5. A double sequence x is divergent in the Pringsheim sense (P-divergent) provided that x does not converge in the Pringsheim sense (P-convergent).

Remark 2.2. Definition 2.5 can also be stated as follows: a double sequence x is P-divergent provided that either x contains at least two subsequences with distinct finite limit points or x contains an unbounded subsequence. Also note that, if x contains an unbounded subsequence then x also contains a definite divergent subsequence.

Remark 2.3. For an ordinary single-dimensional sequence, any sequence is a subsequence of itself. This, however, is not the case in the 2-dimensional plane, as illustrated by the following example.

Example 2.1. The sequence

$$x_{n,k} := \begin{cases} 1, & \text{if } n = k = 0, \\ 1, & \text{if } n = 0, k = 1, \\ 1, & \text{if } n = 1, k = 0, \\ 0, & \text{otherwise} \end{cases} \tag{2.2}$$

contains only two subsequences, namely, $[y_{n,k}] = 0$ for each n and k, and

$$z_{n,k} := \begin{cases} 1, & \text{if } n = k = 0, \\ 0, & \text{otherwise;} \end{cases} \tag{2.3}$$

neither subsequence is x.

The following proposition is easily verified, and is worth stating since each single-dimensional sequence is a subsequence of itself. However, this is not the case for double-dimensional sequences.

Proposition 2.1. *The double sequence x is* P*-convergent to L if and only if every subsequence of x is* P*-convergent to L.*

3. Main results. The next result is a "Steinhaus-type" theorem, so named because of its similarity to the Steinhaus theorem in [9] quoted in the introduction.

Theorem 3.1. *If A is an* RH*-regular matrix, then there exists a bounded double sequence x consisting only of 0's and 1's which is not A-summable.*

Proof. Let m_i, n_j, k_i, and l_j be increasing index sequences which we define as follows:

Let $k_0 := l_0 := -1$ and choose m_0 and n_0 such that $m_0, n_0 > B$, where B is defined by RH_6 and RH_2 to imply

$$\left| \sum_{k,l=0}^{\infty,\infty} a_{m_0,n_0,k,l} \right| > \frac{1}{4}, \tag{3.1}$$

whenever $m_0, n_0 > B$.

Also, by $\text{RH}_1, \text{RH}_3, \text{RH}_4$, and RH_5 we choose $k_1 > k_0$ and $l_1 > l_0$ such that

$$\begin{aligned}
\Big|\sum_{k<k_1,l<l_1} a_{m_0,n_0,k,l}\Big| &> 1-\frac{1}{4},\\
\sum_{k\ge k_1,l\ge l_1} |a_{m_0,n_0,k,l}| &< \frac{1}{4},\\
\sum_{k\ge k_1,l<l_1} |a_{m_0,n_0,k,l}| &< \frac{1}{4},\\
\sum_{k<k_1,l\ge l_1} |a_{m_0,n_0,k,l}| &< \frac{1}{4}.
\end{aligned} \tag{3.2}$$

Next use RH_1, RH_2, RH_3, and RH_4 to choose $m_1 > m_0$ and $n_1 > n_0$ such that

$$\begin{aligned}
\sum_{k<k_1,l<l_1} |a_{m_1,n_1,k,l}| &< \frac{1}{9},\\
\sum_{k\le k_1,l\ge l_1} |a_{m_1,n_1,k,l}| &< \frac{1}{9},\\
\sum_{k\ge k_1,l\le l_1} |a_{m_1,n_1,k,l}| &< \frac{1}{9},\\
\Big|\sum_{k,l=0}^{\infty,\infty} a_{m_1,n_1,k,l}\Big| &> 1-\frac{1}{9}.
\end{aligned} \tag{3.3}$$

These inequalities imply

$$\sum_{k>k_1,l>l_1} |a_{m_1,n_1,k,l}| > 1-\frac{4}{9}, \tag{3.4}$$

because

$$\begin{aligned}
\Big|\sum_{k>k_1,l>l_1} |a_{m_1,n_1,k,l}|\Big| \ge &-\sum_{k\le k_1,l\le l_1} |a_{m_1,n_1,k,l}| + 1 - \frac{1}{9}\\
&-\sum_{k\ge k_1,l\le l_1} |a_{m_1,n_1,k,l}|\\
&-\sum_{k\le k_1,l>l_1} |a_{m_1,n_1,k,l}|.
\end{aligned} \tag{3.5}$$

We now choose $k_2 > k_1$ and $l_2 > l_1$ such that

$$\begin{aligned}
\Big|\sum_{k_1<k<k_2,l_1<l<l_2} a_{m_1,n_1,k,l}\Big| &> 1-\frac{4}{9},\\
\sum_{k\ge k_2,l\ge l_2} |a_{m_1,n_1,k,l}| &< \frac{1}{9},\\
\sum_{k_1<k\le k_2,l\ge l_2} |a_{m_1,n_1,k,l}| &< \frac{1}{9},\\
\sum_{k\ge k_2,l_1<l<l_2} |a_{m_1,n_1,k,l}| &< \frac{1}{9}.
\end{aligned} \tag{3.6}$$

In general, having

$$\begin{aligned} m_0 < \cdots < m_{i-1}, \qquad k_0 < \cdots < k_{i-1} < k_i, \\ n_0 < \cdots < n_{j-1}, \qquad l_0 < \cdots < l_{j-1} < l_j, \end{aligned} \tag{3.7}$$

we choose $m_i > m_{i-1}$ and $n_j > n_{j-1}$ such that by RH_1

$$\sum_{k \le k_i, l \le l_j} |a_{m_i,n_j,k,l}| < \frac{1}{(i+2)(j+2)}, \tag{3.8}$$

and by RH_3, RH_4

$$\begin{aligned} \sum_{k \le k_i, l > l_j} |a_{m_i,n_j,k,l}| &< \frac{1}{(i+2)(j+2)}, \\ \sum_{k \ge k_i, l \le l_j} |a_{m_i,n_j,k,l}| &< \frac{1}{(i+2)(j+2)}. \end{aligned} \tag{3.9}$$

In addition, by RH_2

$$\left| \sum_{k,l=0}^{\infty,\infty} a_{m_i,n_j,k,l} \right| > 1 - \frac{1}{(i+2)(j+2)}, \tag{3.10}$$

so

$$\sum_{k > k_i, l > l_j} |a_{m_i,n_j,k,l}| > 1 - \frac{4}{(i+2)(j+2)}. \tag{3.11}$$

We now choose $k_{i+1} > k_i$ and $l_{j+1} > l_j$ such that

$$\begin{aligned} \left| \sum_{k_i < k < k_{i+1}, l_j < l < l_{j+1}} a_{m_i,n_j,k,l} \right| &> 1 - \frac{4}{(i+2)(j+2)}, \\ \sum_{k \ge k_{i+1}, l \ge l_{j+1}} |a_{m_i,n_j,k,l}| &< \frac{1}{(i+2)(j+2)}, \\ \sum_{k_i < k < k_{i+1}, l \ge l_{j+1}} |a_{m_i,n_j,k,l}| &< \frac{1}{(i+2)(j+2)}, \\ \sum_{k \ge k_{i+1}, l_j < l < l_{j+1}} |a_{m_i,n_j,k,l}| &< \frac{1}{(i+2)(j+2)}. \end{aligned} \tag{3.12}$$

Define x as follows:

$$x_{k,l} = \begin{cases} 1, & \text{if } k_{2p} < k < k_{2p+1} \text{ and } l_{2t} < l < l_{2t+1} \text{ for } p,t = 0,1,2,\ldots, \\ 0, & \text{otherwise .} \end{cases} \tag{3.13}$$

Let us label and partition $(AX)_{m_i,n_j}$ as follows:

$$\begin{aligned}(AX)_{m_i,n_j} = & \sum_{0\le k\le k_i, 0\le l\le l_j}^{\alpha_1} + \sum_{0\le k\le k_i, l_{j+1}\le l}^{\alpha_2} + \sum_{k_{i+1}\le k, l_{j+1}\le l}^{\alpha_3} \\ & + \sum_{0\le l\le l_j, k_{i+1}\le k}^{\alpha_4} + \sum_{k_i<k<k_{i+1}, 0\le l\le l_j}^{\alpha_5} + \sum_{l_j<l<l_{j+1}, 0\le k\le k_i}^{\alpha_6} \\ & + \sum_{k_i<k<k_{i+1}, l_{j+1}\le l}^{\alpha_7} + \sum_{l_j<l<l_{j+1}, k_{i+1}\le k}^{\alpha_8} + \sum_{k_i<k<k_{i+1}, l_j<l<l_{j+1}}^{\alpha_9} a_{m_i,n_j,k,l}x_{k,l},\end{aligned} \tag{3.14}$$

where the general term $a_{m_i,n_j,k,l}x_{k,l}$ is the same for each of the nine sums. Note that,

$$\begin{aligned}|\alpha_4+\alpha_5| &\le \frac{1}{(i+2)(j+2)}, \\ |\alpha_2+\alpha_6| &\le \frac{1}{(i+2)(j+2)}.\end{aligned} \tag{3.15}$$

Case 1. If i and j are even, then

$$\left|(AX)_{m_i,n_j}\right| > 1-\frac{1}{(i+2)(j+2)}-|\alpha_1|-\cdots-|\alpha_8| > 1-\frac{7}{(i+2)(j+2)}, \tag{3.16}$$

and the last expression has P-limit 1.

Case 2. If at least one of i and j is odd, then $\alpha_9 = 0$ and

$$\left|(AX)_{m_i,n_j}\right| \le |\alpha_1|+|\alpha_2|+\cdots+|\alpha_8| \le \frac{6}{(i+2)(j+2)}, \tag{3.17}$$

and the last expression of (3.17) has P-limit 0. Thus the P-limit of $(AX)_{m,n}$ does not exist, and we have shown that an RH-regular matrix A cannot sum every double sequence, of 0's and 1's. □

As with the original Steinhaus Theorem [9], we can state the following as an immediate consequence of Theorem 3.1.

Corollary 3.1. *If A is an* RH*-regular matrix, then A cannot sum every bounded double sequence.*

The next result is a "Buck-type" theorem.

Theorem 3.2. *The bounded double complex sequence x is* P*-convergent if and only if there exists an* RH*-regular matrix A such that A sums every subsequence of x.*

Proof. Since every subsequence of a P-convergent sequence x is bounded and P-convergent, and A is an RH-regular matrix, then for such an x there exists an RH-regular matrix A such that $S''\{x\} \subseteq C''_A$.

Conversely, we use an adaptation of Buck's proof [1] to show that if A is any

RH-regular matrix and $x \notin C''$ then there exists a subsequence $y \in S''\{x\}$ such that $Ay \notin C''$.

Note that every subsequence of x is bounded and $x \notin C''$, which implies that x has at least two distinct Pringsheim limit points, say α and β. Thus there exist increasing index sequences $\{n_j\}$ and $\{k_i\}$ such that $\limsup x_{n_i,k_i} = \alpha$ and $\liminf x_{n_i,k_i} = \beta$ with $\alpha \neq \beta$.

Now define

$$y = \frac{x-\beta}{\alpha-\beta} \tag{3.18}$$

which yields $\limsup y_{n_i,k_i} = 1$ and $\liminf y_{n_i,k_i} = 0$. As a result there exist two disjoint pairs of index sequences $\{\bar{n}^i_j, \bar{k}^i_j\}$ and $\{\nu^i_j, k^i_j\}$ such that the sequences $\bar{y}_1$ and $\bar{y}_2$ constructed using $\{\bar{n}^i_j, \bar{k}^i_j\}$ and $\{\nu^i_j, k^i_j\}$, respectively, have P-limits 1 and 0, respectively. Let

$$y^*_{n,k} := \begin{cases} 1, & \text{if } n = \bar{n}^i_j, k = \bar{k}^i_j, \\ 0, & \text{if } n = \nu^i_j, k = k^i_j, \\ y, & \text{otherwise.} \end{cases} \tag{3.19}$$

Hence, $\{y^*_{n,k}\}$ contains a subsequence $\{\bar{y}^*_{n,k}\}$ with infinitely many 0's and 1's, along its diagonal. This implies that $S''\{\bar{y}^*\}$ contains all sequences of 0's and 1's. Thus by Theorem 3.1, there exists $\tilde{y}^* \in S''\{\bar{y}^*\}$ such that $A\tilde{y}^* \notin C''$. Also, P-lim$(y - y^*)_{i,j} = 0$. We now select a subsequence $\{\tilde{y}_{i,j}\}$ of $\{y_{i,j}\}$ with terms satisfying $\limsup_i y_{n_i,k_i} = 1$ and $\liminf_i y_{n_i,k_i} = 0$ corresponding to the 0's and 1's, respectively of $\{\tilde{y}^*_{i,j}\}$. Therefore P-lim$(\tilde{y} - \tilde{y}^*)_{i,j} = 0$ and $\tilde{y}_{i,j} - \tilde{y}^*_{i,j}$ is bounded. By the linearity and regularity of $A, A(\tilde{y} - \tilde{y}^*)_{i,j} = (A\tilde{y})_{i,j} - (A\tilde{y}^*)_{i,j}$ and P-lim$A(\tilde{y} - \tilde{y}^*)_{i,j} = 0$. Now since $A\tilde{y}^* \notin C''$, it follows that $A\tilde{y} \notin C''$; and since $\tilde{y} = \tilde{x} - \beta/\alpha - \beta$, we have $A\tilde{x} \notin C''$. □

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