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ON THE NUMERICAL TREATMENT OF THE CONTACT PROBLEM

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ABSTRACT. The problem of the contact of two elastic bodies of arbitrary shape with a kernel in the form of a logarithmic function—which is investigated from Hertz problem—is reduced to an integral equation. A numerical method is adapted to determine the pressure between the two surfaces under certain conditions.

Keywords and phrases. Hertz problem, integral equation, orthogonal polynomials.

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1. Introduction. Many problems of mathematical physics, theory of elasticity, viscodynamics fluid and mixed problems of mechanics of continuous media reduce to a Fredholm integral equation with continuous or discontinuous kernel. Integral equation containing singular kernel appears in studies involving airfoil [3], fracture mechanics contact [18] radiation and molecular conduction [6] and others. Over the past thirty years, substantial progress has been made in developing innovative approximate analytical and purely numerical solution to a large class of Fredholm integral equation with singular kernel. Since the theory of singular integral equations developed by Muskhelishvili [8] has assumed various technique and has increasing important applications in different areas of science. For this aim, many different methods are established by Tricomi [16], Popov [15], Green [10] and others for obtaining the solution of the integral equations analytically. Since closed form solution to these integral equations are generally not available, great attention has been focused on the numerical treatment. The interested reader should consult the fine exposition by Golberg [8], Linz [11], Atkinson [4], Delves and Mohamed [5]. Since the Fredholm integral equation of the second kind with Cauchy kernel plays an important rule in applied mathematics and physics, so many different numerical solutions are obtained. For example in [7] Gerasoulis used a piecewise quadratic polynomials in the solution of the singular integral equation. As the same way of Gerasoulis, Miller and Keer [12] obtained the solution of the integral equation with Cauchy kernel. In [17] Venturino used Galerkin method to obtain the singular integral equation of the second kind with Cauchy kernel. In [6] Frankel used a Galerkin approach for solving the integro-differential equation with Cauchy kernel.

In this paper, a numerical method is used to obtain the potential function of a Fredholm integral equation of the second kind with Cauchy kernel. Firstly, we remove the singularity, secondly the solution is expanded in terms of the orthogonal polynomials (we consider the Legendre's polynomial as an example). The solution of the problem reduces to the solution of a linear system. At the end, we give a numerical application to test our method.

2. Formulation of the contact problem. Consider the semi-symmetric problem [14], when the tangent force, $t(x)$, is related with the normal pressure, $p(x)$, in the contact region of the two surfaces, by the relation

$$t(x) = kP(x). \tag{2.1}$$

Also, the normal stress, τ_{xy}, with the tangent stress, σ_y, satisfy the relation

$$\tau_{xy} = k\sigma_y, \tag{2.2}$$

where k is the friction coefficient.

For the displacement components v_i^* $(i = 1,2)$ in the y-direction, we have the relation [15]

$$\frac{dv_1^*}{dx} = \frac{t(x)}{G_1}, \qquad \frac{dv_2^*}{dx} = \frac{t(x)}{G_2}, \tag{2.3}$$

where G_1 and G_2 are the displacement compressible materials of two surfaces $f_1(x)$ and $f_2(x)$, respectively.

It is known that [3] such problem reduces to the following integral equation:

$$k_1\frac{G_1+G_2}{G_1G_2}\int_0^x \phi(t)\,dt + (\nu_1+\nu_2)\int_1^1 k\left(\frac{x-y}{\lambda}\right)\phi(t)\,dt = \delta - f_1(x) - f_2(x), \quad \lambda \in [0,\infty],$$
$$k(t) = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\tan hu}{u}e^{iut}\,du \tag{2.4}$$

under the condition

$$\int_{-1}^{1}\phi(y)\,dy = p < \infty, \quad \phi(-1) = \phi(1) = 0, \quad (p \text{ is constant}), \tag{2.5}$$

where $\phi(t)$ is the unknown potential function which is continuous through the interval of integration $[-1,1]$, the contact domain between the two surfaces $f_i(x)$ $(i = 1,2)$, δ is the rigid displacement under the action of a force P, k_1 is a physical constant, $k(t)$ is the discontinuous kernel of the problem with singularity at the point $x = y$, and $\nu_i = (1-\mu_i^2)/(\pi E_i)$ $(i = 1,2)$ where μ_i are the Poisson's coefficients and E_i are the coefficients of Young.

As in [15], the kernel can be written in the following form

$$k(t) = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\tan hu}{u}e^{iut}du = -\ln\left|\tan h\frac{\pi t}{4}\right|. \tag{2.6}$$

If $\lambda \to \infty$ and the term $(x - y/\lambda)$ is very small, so that it satisfies the condition $\tan hz \simeq z$, then we have

$$\ln\left|\tan h\frac{\pi t}{4}\right| = \ln t - d \quad \left(d = \ln\frac{4\lambda}{\pi}\right). \tag{2.7}$$

Hence, equation (2.4) with the aid of equation (2.7) can be adapted in the form

$$\int_0^x \phi(t)dt + \nu\int_{-1}^{1}[-\ln|y-x| + d]\phi(y)\,dy = f^*(x), \tag{2.8}$$

where,

$$\nu = \frac{(\nu_1+\nu_2)G_1G_2}{k_1(G_1+G_2)}, \qquad f^*(x) = \frac{[\delta - f_1(x) - f_2(x)]G_1G_2}{k[G_1+G_2]}. \tag{2.9}$$

Differentiating equation (2.8) with respect to x, we have

$$\phi(x) + \nu \int_{-1}^{1} \frac{\phi(y)}{y-x}\,dy = f(x) \quad \left(f(x) = \frac{df^*(x)}{dx}\right). \tag{2.10}$$

Equation (2.10) represents a Fredholm integral equation of the second kind with Cauchy kernel which will be solved under the condition (2.5).

Here $\int$ denotes integration with Cauchy principal value sense. We suppose that $\phi(x)$, $x\varepsilon[-1,1]$ is continuous and satisfies the normality condition

$$\left[\int_{-1}^{1} |\phi(y)|^2\,dy\right]^{1/2} \le A\|\phi\|_2, \tag{2.11}$$

where $\| \, \|_2$ denotes the L_2 norm and A is a constant. Moreover, the potential function $\phi(x)$ satisfies the Lipschitz condition with respect to the second argument. Then $\int (\phi(y))/(y-x)\,dy$ exists in the Cauchy principal value sense. It is not difficult to prove the continuity and the normality of the integral operator

$$K\phi = \int_{-1}^{1} \frac{\phi(y)}{y-x}\,dy. \tag{2.12}$$

In the special case $G_1+G_2 = 0$, $f_2(x) = 0$ we have the Fredholm integral equation of the first kind with logarithmic kernel, under the condition (2.5). Abdou and Hassan [2] used potential theory to obtain the eigenvalues and eigenfunctions of the problem. Also, Abdou and Ezz-Eldin [1] used Krein's method to solve the same problem.

3. Solution of the problem. In this section, we will solve equation (2.10) under condition (2.5). In order to reach to our goal the singularity of the integral of equation (2.10) will be weakened as follows:

$$\int_{-1}^{1} \frac{\phi(y)}{y-x}\,dy = \int_{-1}^{1} \frac{\phi(y)-\phi(x)}{y-x}\,dy + \phi(x)\int_{-1}^{1} \frac{1}{y-x}\,dy. \tag{3.1}$$

The first of two right integrals is regular and it will be evaluated later while the second integral is evaluated in [9], it is equal to $-\log(1+x)/(1-x)$. Therefore equation (2.10) becomes

$$\phi(x) + \nu \int_{-1}^{1} \frac{\phi(y)-\phi(x)}{y-x}\,dy - \nu\phi(x)\log\frac{1+x}{1-x} = f(x), \quad -1 < x < 1. \tag{3.2}$$

Assume the unknown function, $\phi(x)$, can be expanded in terms of a series of Legendre's polynomials:

$$\phi(x) = \sum_{j=0}^{\infty} a_j P_j(x) \tag{3.3}$$

(one may use any other orthogonal polynomials expansion).

From condition (2.5), we obtain $a_0 = p$, the rest of the coefficients a_j, $j = 1,2,\ldots$ are to be determined. The Rodrigues' formula of the Legendre polynomial $P_j(x)$ of degree j is given by [9]:

$$P_j(x) = \sum_{k=0}^{[j/2]} \alpha_k x^{j-2k}, \tag{3.4}$$

where

$$\alpha_k = \frac{(-)^k(2j-2k)!}{2jk!(j-k)!(j-2k)!}. \tag{3.5}$$

From which we obtain

$$\int_{-1}^{1} \frac{P_j(y)-P_j(x)}{y-x}\,dy = \sum_{k=0}^{[(j-1)/2]} \alpha_k \sum_{l=0}^{j-2k-1} x^l \int_{-1}^{1} y^{j-2k-1-l}\,dy, \tag{3.6}$$

and therefore

$$\int_{-1}^{1} \frac{P_j(y)-P_j(x)}{y-x}\,dy = \sum_{k=0}^{[(j-1)/2]} \sum_{l=0}^{j-2k-1} \gamma_{j,k,l} x^l, \tag{3.7}$$

where

$$\gamma_{j,k,l} = \frac{\alpha_k\left[1-(-)^{j-l}\right]}{(j-2k-l)}. \tag{3.8}$$

Using equations (3.3) and (3.7), equation (3.2) becomes:

$$\left(1-\nu\log\frac{1+x}{1-x}\right)\sum_{j=0}^{\infty} a_j P_j(x) + \nu \sum_{j=1}^{\infty} a_j \sum_{k=0}^{[(j-1)/2]} \alpha_k \sum_{j=0}^{j-2k-1} \gamma_{j,k,l} x^l = f(x) \tag{3.9}$$

for $-1 < x < 1$.

Multiply both sides of (3.9) by x^{i-1} for $i = 1,2,\ldots,N-1,N$, and then integrating the resultant over the interval $[-1,1]$, we get

$$\begin{aligned} &\sum_{j=1}^{\infty} a_j \int_{-1}^{1} \left(1-\nu\log\frac{1+x}{1-x}\right) x^{i-1} P_j(x)\,dx \\ &\quad + \nu \sum_{j=1}^{\infty} a_j \sum_{k=0}^{[(j-1)/2]} \alpha_k \sum_{l=0}^{j-2k-1} \gamma_{j,k,l} \int_{-1}^{1} x^{l+i-1}\,dx = \int_{-1}^{1} x^{i-l} f(x)\,dx, \end{aligned} \tag{3.10}$$

the term-by-term integration is justified by the uniform convergence of each of the previous three series of the left side of the previous equation in the interval [-1,1] and

$$\left|x^i P_j(x)\right| \le \left|x^i\right| \le 1, \quad |x| < 1. \tag{3.11}$$

For $|x| < 1$, we can assume

$$\log\frac{1+x}{1-x} \simeq 2x, \tag{3.12}$$

and so equation (3.10) will take the form

$$\sum_{j=1}^{\infty} a_j \left\{ \int_{-1}^{1} (1-2\nu x) x^{i-1} P_j(x)\,dx + \nu \sum_{k=0}^{[(j-1)/2]} \alpha_k \sum_{l=0}^{j-2k-1} \frac{2[1-(-)^{j-l}]}{(j-2k-l)(l+j)} \delta_{l+j-1} \right\}$$
$$= \int_{-1}^{1} x^{i-1} f(x)\,dx - 2p\left(\frac{\delta_{i-1}}{i} - \frac{2\nu\delta_i}{i+1}\right), \quad \text{for } i = 1,2,\dots,N-1,N, \tag{3.13}$$

where

$$\delta_c = \begin{cases} 1, & c \text{ even}, \\ 0, & c \text{ odd}. \end{cases} \tag{3.14}$$

And we will evaluate the integral of the left side of this equation by using the famous Rodrigues' formula. In fact this integral is equal to

$$\int_{-1}^{1} x^i P_j(x)\,dx = \sum_{k=0}^{[j/2]} \alpha_k \frac{2\delta_{i,j}}{j+i-2k+1}, \tag{3.15}$$

where

$$\delta_{c,d} = \begin{cases} 1, & c+d \text{ even and } c \ge d, \\ 0, & \text{otherwise}. \end{cases} \tag{3.16}$$

From which the solution of equation (2.10) is the solution of the following linear system:

$$\sum_{j=1}^{\infty} a_j \left\{ \sum_{k=0}^{[j/2]} \left(\frac{2\alpha_k \delta_{i-1,j}}{j+i-2k} - \frac{4\nu\alpha_k\delta_{i,j}}{j+i-2k+1} \right) + \sum_{k=0}^{[(j-1)/2]} \sum_{l=0}^{j-2k-1} \frac{2\nu\alpha_k[1-(-)^{j-1}]}{(j-2k-l)(l+i)} \delta_{l+i-1} \right\}$$
$$= \int_{-1}^{1} x^{i-1} f(x)\,dx - 2p\left(\frac{\delta_{i-1}}{i} - \frac{2\nu\delta_i}{i+1}\right), \tag{3.17}$$

for $i = 1,2,\dots,N-1,N$.

If we truncate the infinite series of the left side of the previous linear system to the first N terms, this linear system will take the form

$$\sum_{j=1}^{N} c_{ij} a_j = b_i, \quad i = 1,2,\dots,N-1,N, \tag{3.18}$$

where

$$c_{ij} = \sum_{k=0}^{[j/2]} \left(\frac{2\alpha_k \delta_{i-1,j}}{j+i-2k} - \frac{4\nu\alpha_k\delta_{i,j}}{j+i-2k+1} \right) + \sum_{k=0}^{[(j-1)/2]} \sum_{l=0}^{j-2k-1} \frac{2\nu\alpha_k[1-(-)^{j-1}]}{(j-2k-l)(l+i)} \delta_{l+i-1} \tag{3.19}$$

and

$$b_i = \int_{-1}^{1} x^{i-1} f(x)\,dx - 2p\left(\frac{\delta_{i-1}}{i} - \frac{2\nu\delta_i}{i+1}\right). \tag{3.20}$$

TABLE 4.1. The coefficients a_j, $j = 1, 2, \ldots, 20$ (ordered in rows).

−0.796513983	1.282960283	−1.39797130	1.456975757	−1.549072869
1.479876614	−1.567379746	1.417647537	−1.559090422	1.178337053
−1.392267336	2.037120656	−1.596255523	−2.924756126	−50248330647
8.38415880	6.333866610	1.704701286	−0.252825184	−1.196720603

4. Numerical example. The solution of the integral equation (3.2) depends on the Cauchy kernel and the two surfaces $f_1(x)$ and $f_2(x)$. We can expand each of these two functions in Macklaurin expansion near $x = 0$ where the initial points and the tangent points of the surfaces are in contact with the origin O. For that reason, we can assume the function $f(x)$ is a polynomial (see the definition of the function $f(x)$, equation (2.10)).

If we take only the quadratic term in the Maclorian expansion of the function $f_1(x) + f_2(x)$, then we may assume $f(x) = x$. Also, take $p = 0.8$, $\nu = 0.25$, and $N = 10$. The pressure between the two surfaces in this case is given by

$$\phi(x) = \sum_{j=0}^{10} a_j P_j(x), \tag{4.1}$$

where, $P_j(x)$ is Legendre's polynomial of degree j and the coefficients a_j are the solution of the linear system (3.18), we used Maple V to solve such system. These coefficients are tabulated in Table 4.1.

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