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A GENERALIZATION OF A THEOREM OF FAITH AND MENAL AND APPLICATIONS

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Abstract. In 1995, Faith and Menal have established the V-ring theorem which gives a characterization of a V-ring. In this paper, we generalize this theorem to V-modules and consider some applications for Noetherian self-cogenerators.

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1. Preliminaries. Throughout this paper, R denotes an associative ring with identity and all modules considered are unitary right R-modules. Homomorphisms are written on the side opposite to that of scalars. For any module M, the sum of all simple submodules of M is called a *socle* of M and is denoted by $\mathrm{Soc}(M)$. Dually, the intersection of all maximal submodules of M is called a *radical* of M and is denoted by $\mathrm{Rad}(M)$. $(R)_n$ denotes the $n\times n$ matrix ring over R. Let M be a module. An *M-generated module* is a module which is isomorphic to a factor module of $M^{(I)}$ for some index set I. We denote by $\sigma[M]$ the full subcategory of Mod-R whose objects are all submodules of M-generated modules, and by $E_M(N)$ the *M-injective hull* of a module N in $\sigma[M]$ which is the trace of M in $E(M)$, where $E(M)$ indicates the injective hull of M, that is $E_M(N)=\sum\{f(M) : f\in \mathrm{Hom}_R(M,E(N))\}$ in $\sigma[M]$ (see Wisbauer [9, 17.9, (2)]). A module M is called a *V-module* if every proper submodule of M is an intersection of maximal submodules of M or, equivalently, if every simple module (in $\sigma[M]$ or Mod-R) is M-injective (see, e.g., Wisbauer [9, 23.1]). A ring R is called a *right V-ring* if R is a V-module when considered as a right module over itself, i.e., every simple module is injective. For notation, definitions and, familiar results concerning the ring theory we mainly follow Anderson and Fuller [2] and Wisbauer [9].

2. A generalization of a theorem of Faith and Menal. Let M,E be modules, and $M_E^*=\mathrm{Hom}_R(M,E)$. For each subset Z of M_E^* and each subset X of M, the right annihilator in M is denoted by $r_M(Z)$, and the left annihilator in M_E^* is denoted by $\ell_{M_E^*}(X)$, that is,

$$r_M(Z)=\{m\in M : Zm=0\},\qquad \ell_{M_E^*}(X)=\{f\in M_E^* : fX=0\}. \tag{2.1}$$

In [6], Faith and Menal showed that a ring R is a right V-ring if and only if there exists a semisimple module W such that $I=r_R\ell_W(I)$ for every right ideal I of R. In this case, we say that W satisfies the double annihilator condition (d.a.c.) with respect to right ideals. This characterization of a V-ring by the existence of a duality between the right ideals via annihilation and submodules of a semisimple module is called the *V-ring theorem*. A ring R is called a *right Johns ring* if R is right Noetherian and satisfies that any right ideal is a right annihilator ideal. It is known that a right Johns ring is not right Artinian (see Faith and Menal [4]). If a ring R is right Johns, then $\tilde{I} = r_{R/J(R)}\ell_{\mathrm{Soc}(R)}(\tilde{I})$ for any right ideal $\tilde{I}$ of $R/J(R)$, that is $R/J(R)$ is a right V-ring by the V-ring theorem, where $J(R)$ denotes the Jacobson radical of R (see Faith and Menal [6]). We begin with the following theorem.

Theorem 2.1. *Let M be a module. Then the following are equivalent:*
(1) *M is a V-module;*
(2) *there exists a semisimple module W satisfying $N = r_L\ell_{L^*_W}(N)$ for any module L in* Mod-R *and any submodule N of L such that L/N is in $\sigma[M]$;*
(3) *there exists a semisimple module W' in $\sigma[M]$ satisfying $N = r_L\ell_{L^*_{W'}}(N)$ for any module L in $\sigma[M]$ and any submodule N of L.*

Proof. (1)⇒(2), (1)⇒(3). Let $\{S_i\}_{i\in\Omega}$ be an irredundant set of representatives of the simple modules in $\sigma[M]$. Then, $\bigoplus_{i\in\Omega} E_M(S_i)$ is the minimal M-injective cogenerator of $\sigma[M]$ (see Wisbauer [9, p. 143]).

Since M is a V-module, $E_M(S_i) = S_i$ for each $i \in \Omega$ and, hence, $\bigoplus_{i\in\Omega} S_i$ is a semisimple cogenerator of $\sigma[M]$. Hence, $\bigoplus_{i\in\Omega} S_i$ cogenerates L/N for any module L and any submodule $N \subseteq L$ such that L/N in $\sigma[M]$. By Albu and Năstăsescu [1, Prop. 3.5], $\bigoplus_{i\in\Omega} S_i$ cogenerates the factor module L/N if and only if $N = r_L\ell_{L^*_{\oplus_{i\in\Omega} S_i}}(N)$. Now, the proof of (1)⇒(2) is clear.

Since $\bigoplus_{i\in\Omega} S_i$ is in $\sigma[M]$ and, for any module L in $\sigma[M]$, each factor module of L belongs to $\sigma[M]$, the implication (1)⇒(3) also follows from the proof above.

(2)⇒(1), (3)⇒(1). For a semisimple module W satisfying condition (2), since each factor module of M belongs to $\sigma[M]$, we see that $N = r_M\ell_{M^*_W}(N)$ holds for any submodule N of M. Hence, $M/N \to W^{\ell_{M^*_W}(N)}, m+N \mapsto (f(m))_{f\in\ell_{M^*_W}(N)}$ is an R-monomorphism. This readily implies that $\mathrm{Rad}(M/N) = 0$. Hence, N is an intersection of maximal submodules of M. Thus, M is a V-module. For a semisimple module W' satisfying condition (3), it also follows from the same argument above that M is a V-module. □

Corollary 2.2. *Let M be a module. Then the following statements are equivalent:*
(1) *M is a V-module;*
(2) *there exists a semisimple module W satisfying $I = r_R\ell_W(I)$ for any right ideal I of R such that R/I is in $\sigma[M]$;*
(3) *there exists a semisimple module W' in $\sigma[M]$ satisfying $N = r_M\ell_{M^*_{W'}}(N)$ for any submodule N of M.*

In this case, W and W' cogenerate any module in $\sigma[M]$.

Proof. (1)⇒(2), (1)⇒(3). These are obvious by Theorem 2.1.

(3)⇒(1). Follows immediately from the same argument of (3)⇒(1) in the proof of Theorem 2.1.

(2)⇒(1). Let S be any simple module in $\sigma[M]$. To show that S is M-injective, we need to show that S is N-injective for every cyclic submodule N of M by Wisbauer [9, 16.3, (b)]. So, let N be a cyclic submodule of M and let f be a nonzero R-homomorphism from a submodule N' of N to S. Since N is cyclic, $N \cong R/I$ for some right ideal I of R and, hence, $N' \cong L/I$ for some right ideal L of R. Therefore, $\mathrm{Ker}(f) \cong L'/I$ for some right ideal $L' \subset L$ of R. Since N, $\mathrm{Ker}(f)$ are in $\sigma[M]$ and since $\sigma[M]$ is closed under cokernels, R/L' is in $\sigma[M]$. The hypothesis implies that $L' = r_R\ell_W(L')$. By [1, Prop. 3.5], there is an exact sequence $0 \to R/L' \to W^Y$ for some set Y. This readily implies that $\mathrm{Rad}(R/L') = 0$. Then since L' is an intersection of maximal right ideals, there is a maximal right ideal K of R such that $K \supseteq L'$ but $K \not\supseteq L$. Since $N'/\mathrm{Ker}(f) \cong L/L'$ is simple, it follows that $L \cap K = L'$. Then $R/I/K/I \cong R/K = (L+K)/K \cong (L/L\cap K) = L/L' \cong N'/\mathrm{Ker}(f) \cong S$ and, therefore, f can be extended to an $\bar{f}$ in $\mathrm{Hom}_R(N,S)$. Hence, S is N-injective and M is a V-module.

Finally, we show that a semisimple module W satisfying condition (2) and a semisimple module W' satisfying condition (3) cogenerate any module in $\sigma[M]$. For any maximal right ideal I with R/I in $\sigma[M]$, we observe that $I = r_R\ell_W(I)$ holds. Thus, it follows, by almost same argument in the proof of the corollary in Faith and Menal [6], that W satisfying condition (2) cogenerates any module in $\sigma[M]$. Next, since $E_M(S) = S$ for any simple module S in $\sigma[M]$, $f(M) = S$ for some $f \in \mathrm{Hom}_R(M,S)$ and, hence, $M/\mathrm{Ker}(f) \cong S$. Then since W' satisfies the d.a.c. with respect to the submodules of M, $\mathrm{Ker}(f) = r_M\ell_{M^*_{W'}}(\mathrm{Ker}(f))$. Since $\mathrm{Ker}(f)$ is maximal, $\mathrm{Ker}(f) = r_M(g) = \mathrm{Ker}(g)$ for some $g \in M^*_{W'}$. Therefore, W' contains a copy of S. This implies that W' satisfying condition (3) cogenerates any module in $\sigma[M]$. □

Remark 2.3. Let M be a module. If there exists a semisimple module W, which need not be in $\sigma[M]$, such that W satisfies the d.a.c. with respect to any submodule of M, then it is easy to deduce from the argument of the proof of (2)⇒(1) and (3)⇒(1) in Theorem 2.1 that M is a V-module.

Proposition 2.4. *Let M be a module. If M contains a copy of each simple factor module of M, then the following statements are equivalent:*

(1) $M/\mathrm{Rad}(M)$ *is a V-module;*

(2) $\mathrm{Soc}(M)$ *cogenerates any module in* $\sigma[M/\mathrm{Rad}(M)]$*;*

(3) $\tilde{I} = r_{R/J(R)}\ell_{\mathrm{Soc}(M)}(\tilde{I})$ *for any right ideal $\tilde{I}$ of $R/J(R)$ such that $(R/J(R))/\tilde{I}$ is in* $\bar{\sigma}[M/\mathrm{Rad}(M)] = \sigma[M/\mathrm{Rad}(M)] \cap (\mathrm{Mod}\text{-}R/J(R))$.

Proof. (1)⇒(2). Let $\{S\}_{i\in\Omega}$ be an irredundant set of representatives of the simple R-modules in $\sigma[M/\mathrm{Rad}(M)]$. Since $M/\mathrm{Rad}(M)$ is a V-module, by Wisbauer [9, p. 143], we know that $\bigoplus_{i\in\Omega} S_i$ cogenerates any module in $\sigma[M/\mathrm{Rad}(M)]$. So, it suffices to show that $\mathrm{Soc}(M)$ contains a copy of S_i for each $i \in \Omega$. Since $E_{M/\mathrm{Rad}(M)}(S_i) = S_i, f(M/\mathrm{Rad}(M)) = S_i$ for some $f \in \mathrm{Hom}_R(M/\mathrm{Rad}(M), S_i)$. Clearly, S_i is a simple homomorphic image of M. Thus, by hypothesis, there exists an exact sequence $0 \to S_i \to \mathrm{Soc}(M)$. Obviously, it follows that $\mathrm{Soc}(M)$ cogenerates any module in $\sigma[M/\mathrm{Rad}(M)]$.

(2)⇒(3). We note that any module in $\bar{\sigma}[M/\mathrm{Rad}(M)]$ belongs to $\sigma[M/\mathrm{Rad}(M)]$. Since $\mathrm{Soc}(M)$ cogenerates any module in $\bar{\sigma}[M/\mathrm{Rad}(M)]$, again by virtue of [1, Prop. 3.5], we have $\tilde{I} = r_{R/J(R)}\ell_{\mathrm{Soc}}(M)(\tilde{I})$ for every right ideal $\tilde{I}$ of $R/J(R)$ such that $R/J(R)/\tilde{I}$ in $\bar{\sigma}[M/\mathrm{Rad}(M)]$.

(3)⇒(1). Note that $M/\mathrm{Rad}(M)$ is a V-module as a right $R/J(R)$-module if and only if $M/\mathrm{Rad}(M)$ is a V-module as a right R-module. Since $M/\mathrm{Rad}(M)_{R/J(R)}$ is a V-module by Corollary 2.2, $M/\mathrm{Rad}(M)_R$ is a V-module. □

Recall that a ring R is a *right Kasch ring* if any simple right R-module is isomorphic to a minimal right ideal of R. Since a ring R is right Kasch if and only if every maximal right ideal of R is a right annihilator ideal (see, e.g., Faith [3, p. 37]), we observe that a right Johns ring is right Kasch.

Corollary 2.5. *If a ring R is right Kasch, then the following statements are equivalent:*

(1) $R/J(R)$ *is a right V-ring;*

(2) $\mathrm{Soc}(R)$ *cogenerates any module in* Mod-$R/J(R)$*;*

(3) $\tilde{I} = r_{R/J(R)}\ell_{\mathrm{Soc}(R)}(\tilde{I})$ *for every right ideal $\tilde{I}$ of $R/J(R)$.*

3. Applications. A module M is called a *self-generator* if M generates every submodule of M. Dually, a module M is called a *self-cogenerator* if M cogenerates every factor module of M. By Albu and Năstăseacu [1, Prop. 3.5], M is a self-cogenerator if and only if $N = r_M\ell_\Lambda(N)$ for any submodule N of M, where $\Lambda = \mathrm{End}(M_R)$. In particular, R_R is a self-cogenerator if and only if $I = r_R\ell_R(I)$ for any right ideal I of R.

Theorem 3.1. *Let M be a self-cogenerator and let $\Lambda = \mathrm{End}(M_R)$. If there exists a (Λ,R)-bimodule $W \subseteq \mathrm{Soc}(M_R)$ such that $M_W^* = \ell_\Lambda(X)$ for some subset X of M, then $\bar{M} = M/r_M(M_W^*)$ is a V-module.*

Proof. By virtue of Remark 2.3, we need to prove that $N/r_M(M_W^*) = r_{\bar{M}}\ell_{\bar{M}_W^*}(N/r_M(M_W^*))$ for every submodule $N \supseteq r_M(M_W^*)$ of M. Applying the W-dual functor $\mathrm{Hom}_R(-,W)$ to the natural exact sequence $M \to \bar{M} \to 0$, we get that the dual sequence $0 \to \bar{M}_W^* \to M_W^*$ is exact. Since $\ell_\Lambda r_M(M_W^*) = \ell_\Lambda r_M \ell_\Lambda(X) = \ell_\Lambda(X) = M_W^*$ by hypothesis, we have $M_W^* \cong \bar{M}_W^*$ as an abelian group. Since M is a self-cogenerator, there exists a subset $\{g_i\}_{i\in I} \subseteq \Lambda$ such that $N = r_M(\{g_i\}_{i\in I})$. If we take the left annihilator in Λ for $r_M(M_W^*) \subseteq N$, we have $\{g_i\}_{i\in I} \subseteq \ell_\Lambda(N) \subseteq \ell_\Lambda r_M(M_W^*) = M_W^*$. Since $M_W^* \cong \bar{M}_W^*$ by the natural way, so that $\{\bar{g}_i\}_{i\in I} \subseteq \bar{M}_W^*$ follows, where $\bar{g}_i : \bar{M} \to W$ denotes the R-homomorphism induced by g_i for each $i \in I$. Thus, we obtain that $\{\bar{g}_i\}_{i\in I} \subseteq \ell_{\bar{M}_W^*}(N/r_M(M_W^*))$. So, if we note that $r_{\bar{M}}(\{\bar{g}_i\}_{i\in I}) = r_M(\{g_i\}_{i\in I})/r_M(M_W^*)$, then we have

$$r_{\bar{M}}\ell_{\bar{M}_W^*}\left(\frac{N}{r_M(M_W^*)}\right) \subseteq r_{\bar{M}}(\{\bar{g}_i\}_{i\in I}) = \frac{r_M(\{g_i\}_{i\in I})}{r_M(M_W^*)} = \frac{N}{r_M(M_W^*)}. \tag{3.1}$$

Since the reverse inclusion is easily verified, this completes the proof. □

Observe that a right Johns ring is a trivial Noetherian self-cogenerator. Next, we consider a nontrivial module which is a Noetherian self-cogenerator. It is known that the class of right Johns rings is not Morita stable (see Faith and Menal [5, Rem. 3.7]). A ring R is called a *strongly right Johns ring* if $(R)_n$ is right Johns for all positive integers n. However, it is not known if a strongly right Johns ring must be quasi-Frobenius, equivalently, right Artinian (cf. Faith and Menal [5]). Using a right Johns ring and a strongly right Johns ring, we construct Noetherian self-cogenerators. Let $n > 0$, $S = (R)_n$ and $P = R^{(n)}$. Consider the functor $H = \mathrm{Hom}_R(P,-) : \text{Mod-}R \to \text{Mod-}S$. We note that the functor $H = \mathrm{Hom}_R(P,-) : \text{Mod-}R \to \text{Mod-}S$ is an equivalence.

Example 1. Suppose that R is a strongly right Johns ring and consider $P = R^{(n)}$. Since $H(R)^n \cong H(P) \cong S$, every factor module of P_R is cogenerated by R if and only if every factor module of S_S is cogenerated by $H(R)$ if and only if every factor module of S_S is cogenerated by S. Thus, P_R gives an example of a Noetherian self-cogenerator.

Example 2. Suppose that R is a right Johns ring and consider $P = R^{(n)}$ as a right S-module by the usual way. By Anderson and Fuller [2, Prop. 21.7], each submodule of $H(R)_S$ is of the form $\operatorname{Im} H(g)$ for some submodule I of R_R and the inclusion map $g : I \to R$. Since R_R is a self-cogenerator, $I = r_R \ell_R(I)$ holds for any right ideal I of R. By Kurata and Hashimoto [8, Lem. 1.19], we have $\operatorname{Im} H(g) = r_{H(R)} \ell_R(\operatorname{Im} H(g))$. Then, $H(R)/\operatorname{Im} H(g) \to H(R)^{\ell_R(\operatorname{Im} H(g))}$, $m + \operatorname{Im} H(g) \mapsto (rm)_{r \in \ell_R(\operatorname{Im} H(g))}$ is an S-monomorphism. Thus, $H(R)_S$ is a self-cogenerator. Since $P_S \cong H(R)_S$ is a natural isomorphism, $H(R)_S$ is a self-cogenerator if and only if P_S is a self-cogenerator. Thus P_S is a self-cogenerator. Since S is right Noetherian, the finitely generated module P_S is right Noetherian. Therefore, P_S gives an example of a Noetherian self-cogenerator.

Proposition 3.2. *If M is a Noetherian projective self-cogenerator, then $\Lambda = \operatorname{End}(M_R)$ is a right Johns ring and $\operatorname{End}(M/\operatorname{Rad}(M)_R)$ is a right V-ring.*

Proof. Suppose that I is any finitely generated right ideal of Λ. Since M is projective, $I = \operatorname{Hom}_R(M, IM)$ by Wisbauer [9, 18.4]. Since M is a self-cogenerator, there is some set Y of Λ such that $IM = r_M(Y)$. Now, it is straightforward to verify that

$$\operatorname{Hom}_R\left(M, r_M(Y)\right) = r_\Lambda(Y). \tag{3.2}$$

This implies that I is a right annihilator ideal. Since M is Noetherian and projective, it follows from Albu and Năstăsescu [1, Prop. 4.12] that Λ is right Noetherian. Hence, Λ is a right Johns ring. Now, by Anderson and Fuller [2, Cor. 17.12], $\operatorname{End}(M/\operatorname{Rad}(M)_R) \cong \Lambda/J(\Lambda)$. Since $\Lambda/J(\Lambda)$ is a right V-ring, $\operatorname{End}(M/\operatorname{Rad}(M)_R)$ is a right V-ring. □

Corollary 3.3. *Let M be a Noetherian projective self-generator and a self-cogenerator, then $M/\operatorname{Rad}(M)$ is a V-module.*

Proof. We note that $M/\operatorname{Rad}(M)$ is projective and a self-generator in Mod-$R/J(R)$. By Proposition 3.2, $\operatorname{End}(M/\operatorname{Rad}(M)_R)$ is a right V-ring and, hence, $\operatorname{End}(M/\operatorname{Rad}(M)_{R/J(R)})$ is a right V-ring. Thus, by Hirano [7, Thm. 3.11], $M/\operatorname{Rad}(M)_{R/J(R)}$ is a V-module and so $M/\operatorname{Rad}(M)_R$ is a V-module. □

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