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A GEOMETRIC CHARACTERIZATION OF FINSLER MANIFOLDS OF CONSTANT CURVATURE $K = 1$

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ABSTRACT. We prove that a Finsler manifold $\mathbb{F}^m$ is of constant curvature $K = 1$ if and only if the unit horizontal Liouville vector field is a Killing vector field on the indicatrix bundle IM of $\mathbb{F}^m$.

Keywords and phrases. Finsler manifold of constant curvature, Killing vector field, indicatrix bundle, horizontal Liouville vector field.

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1. Introduction. The geometry of Finsler manifolds of constant curvature is one of the fundamental subjects in Finsler geometry. Akbar-Zadeh [1] proved that, under some conditions on the growth of the Cartan tensor, a Finsler manifold of constant curvature K is locally Minkowskian if $K = 0$ and Riemannian if $K = -1$. Recently, Bryant [5] has constructed interesting Finsler metrics of positive constant curvature on the sphere S^2. Shen [9] has also investigated the geometric structure of Finsler manifolds of positive constant curvature via the Riemannian Y-metrics. Some special Finsler metrics of constant curvature have been intensively studied by Matsumoto [7, 8], Shibata-Kitayama [10], and Wei [11].

The purpose of the present paper is to obtain a geometric characterization of Finsler manifolds of positive constant curvature. More precisely, we prove that $\mathbb{F}^m = (M, F)$ is a Finsler manifold of constant curvature $K = 1$ if and only if the unit horizontal Liouville vector field $\xi = (y^i/F)\delta/\delta x^i$ is a Killing vector field on the indicatrix bundle IM of $\mathbb{F}^m$. To achieve this result, we consider the Sasaki-Finsler metric G on TM and prove that the linear connection of the Cartan connection on $\mathbb{F}^m$ is just the projection of the Levi-Civita connection ∇ with respect to G on the vertical vector bundle (see Theorem 2.1). This enables us to express the local coefficients of ∇ in terms of the local coefficients of the Cartan connection of $\mathbb{F}^m$ (see Theorem 2.2). Finally, a necessary and sufficient condition for ξ to be a Killing vector field on IM leads to the proof of the main result stated in Theorem 3.3.

2. The Levi-Civita connection with respect to a Sasaki-Finsler metric. In the present section, we show that the linear connection of the Cartan connection is the projection of the Levi-Civita connection with respect to the Sasaki-Finsler metric on the vertical vector bundle. Then we express the local coefficients of the Levi-Civita connection in terms of the local coefficients of the Cartan connection.

Throughout the paper we use the Einstein convention, that is, repeated indices with one upper index and one lower index denotes summation over their range. Also, for any smooth manifold N, we denote by $\mathcal{F}(N)$ the algebra of smooth functions on N and by $\Gamma(E)$ the $\mathcal{F}(N)$-module of smooth sections of a vector bundle E over N. For some Finsler tensor fields we put the index o to denote the contraction by the supporting element y^i, as for example, $T_{io} = T_{ij}y^j$.

Let $\mathbb{F}^m = (M,F)$ be a Finsler manifold, where M is a real m-dimensional smooth manifold and F is the fundamental function of $\mathbb{F}^m$ (see Antonelli-Ingarden-Matsumoto [2, page 36]). Consider $TM^\circ = TM\backslash\{0\}$ and denote by VTM° the vertical vector bundle over TM°, that is, $VTM^\circ = \ker \pi_*$, where π_* is the tangent mapping of the canonical projection $\pi : TM^\circ \longrightarrow M$. We may think of the Finsler metric $g = (g_{ij}(x,y))$, where we set

$$g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j} \tag{2.1}$$

as a Riemannian metric on VTM°. The canonical nonlinear connection $HTM^\circ = (N_i^j(x,y))$ of $\mathbb{F}^m$ is given by

$$N_i^j = \frac{\partial G^j}{\partial y^i}, \tag{2.2a}$$

$$G^j = \frac{1}{4}g^{jh}\left(\frac{\partial^2 F^2}{\partial y^h \partial x^k}y^k - \frac{\partial F^2}{\partial x^h}\right). \tag{2.2b}$$

Then on any coordinate neighborhood $\mathcal{U} \subset TM^\circ$ the vector fields

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}, \quad i \in \{1,\dots,m\}, \tag{2.3}$$

form a basis for $\Gamma(HTM^\circ_{|\mathcal{U}})$. By straightforward calculations using (2.3) we obtain the following Lie brackets:

$$\left[\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right] = G_i{}^k{}_j \frac{\partial}{\partial y^k} \tag{2.4}$$

$$\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] = R^k{}_{ij} \frac{\partial}{\partial y^k}, \tag{2.5}$$

where we set

$$G_i{}^k{}_j = \frac{\partial N_i^k}{\partial y^j} \tag{2.6a}$$

$$R^k{}_{ij} = \frac{\delta N_i^k}{\delta x^j} - \frac{\delta N_j^k}{\delta x^i}. \tag{2.6b}$$

On TM° we consider the almost product structure Q locally given by

$$Q\left(\frac{\partial}{\partial y^i}\right) = \frac{\delta}{\delta x^i} \qquad \text{and} \qquad Q\left(\frac{\delta}{\delta x^i}\right) = \frac{\partial}{\partial y^i}. \tag{2.7}$$

Then by means of the pair (g,Q), we define a Riemannian metric G on TM° by (cf. Bejancu [4, page 42])

$$G(X,Y) = g(vX,vY) + g(QhX,QhY) \quad \forall X,Y \in \Gamma(TTM^\circ), \tag{2.8}$$

where v and h denote the projection morphisms of TTM° on VTM° and HTM°, respectively. Clearly, we have

$$G\left(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\right) = G\left(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\right) = g_{ij}, \qquad G\left(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\right) = 0, \tag{2.9}$$

that is, HTM° and VTM° are complementary orthogonal vector subbundles of TTM° with respect to G. As the Riemannian metric G is of Sasaki type and was obtained from a Finsler metric, we call it the *Sasaki-Finsler metric* on TM°.

The Levi-Civita connection ∇ on TM° with respect to G is given by the well-known formula

$$\begin{aligned} 2G(\nabla_X Y,Z) &= X(G(Y,Z)) + Y(G(X,Z)) - Z(G(X,Y)) \\ &\quad + G([X,Y],Z) + G([Z,X],Y) - G([Y,Z],X), \end{aligned} \tag{2.10}$$

for any $X,Y,Z \in \Gamma(TTM^\circ)$.

On the other hand, the Cartan connection of $\mathbb{F}^m$ is the pair $FC = (HTM^\circ, \nabla^\circ)$, where HTM° is the canonical nonlinear connection given by (2.2) and ∇° is a linear connection on VTM° whose local coefficients ${C_i}^k{}_j$ and ${F_i}^k{}_j$ are given by

$$\nabla^\circ_{\partial/\partial y^j}\frac{\partial}{\partial y^i} = {C_i}^k{}_j\frac{\partial}{\partial y^k}, \tag{2.11a}$$

$${C_i}^k{}_j = \frac{1}{2}g^{kh}\frac{\partial g_{hi}}{\partial y^j} \tag{2.11b}$$

and

$$\nabla^\circ_{\delta/\delta x^j}\frac{\partial}{\partial y^i} = {F_i}^k{}_j\frac{\partial}{\partial y^k}, \tag{2.12a}$$

$${F_i}^k{}_j = \frac{1}{2}g^{kh}\left(\frac{\delta g_{hi}}{\delta x^j} + \frac{\delta g_{hj}}{\delta x^i} - \frac{\delta g_{ij}}{\delta x^h}\right), \tag{2.12b}$$

respectively. The h- and v-covariant derivatives of a Finsler tensor field $T = (T^{j\cdots}_{i\cdots})$ are denoted by $T^{j\cdots}_{i\cdots|k}$ and $T^{j\cdots}_{i\cdots\|k}$, respectively.

In order to get an interrelation between the Levi-Civita connection ∇ and the linear connection ∇° of the Cartan connection we set $G_j = g_{jh}G^h$, and by direct calculations using (2.1) and (2.2b), we deduce that

$$\frac{\partial}{\partial y^k}\left(\frac{\partial G_i}{\partial y^j} - \frac{\partial G_j}{\partial y^i}\right) = \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}. \tag{2.13}$$

Now, we state the following result.

Theorem 2.1. *The linear connection ∇° of the Cartan connection FC is the projection of the Levi-Civita connection ∇ on VTM°, i.e., we have*

$$\nabla^\circ_X Y = v\nabla_X Y, \tag{2.14}$$

for any $X \in \Gamma(TTM^\circ)$ and $Y \in \Gamma(VTM^\circ)$.

Proof. First, we put

$$v\nabla_{\partial/\partial y^j}\frac{\partial}{\partial y^i} = A_i{}^k{}_j\frac{\partial}{\partial y^k} \qquad \text{and} \qquad v\nabla_{\delta/\delta x^j}\frac{\partial}{\partial y^i} = B_i{}^k{}_j\frac{\partial}{\partial y^k}. \tag{2.15}$$

Then in (2.10) we replace (X,Y,Z) in turn by $(\partial/\partial y^j, \partial/\partial y^i, \partial/\partial y^k)$ and $(\delta/\delta x^j, \partial/\partial y^i, \partial/\partial y^k)$ and by using (2.1), (2.4), (2.9), and (2.11b), we obtain

$$A_i{}^k{}_j = C_i{}^k{}_j \tag{2.16}$$

and

$$B_i{}^k{}_j = \frac{1}{2}g^{kh}\left(\frac{\delta g_{hi}}{\delta x^j} + g_{th}G_j{}^t{}_i - g_{ti}G_j{}^t{}_h\right). \tag{2.17}$$

Furthermore, by using (2.2a), (2.3) and (2.13), we derive

$$\begin{aligned}
g_{th}G_j{}^t{}_i - g_{ti}G_j{}^t{}_h &= g_{th}\frac{\partial^2 G^t}{\partial y^i \partial y^j} - g_{ti}\frac{\partial^2 G^t}{\partial y^h \partial y^j} \\
&= \frac{\partial}{\partial y^j}\left(\frac{\partial G_h}{\partial y^i} - \frac{\partial G_i}{\partial y^h}\right) - N_i^t\frac{\partial g_{th}}{\partial y^j} + N_h^t\frac{\partial g_{ti}}{\partial y^j} \\
&= \left(\frac{\partial g_{hj}}{\partial x^i} - N_i^t\frac{\partial g_{hj}}{\partial y^t}\right) - \left(\frac{\partial g_{ij}}{\partial x^h} - N_h^t\frac{\partial g_{ij}}{\partial y^t}\right) \\
&= \frac{\delta g_{hj}}{\delta x^i} - \frac{\delta g_{ij}}{\delta x^h}.
\end{aligned} \tag{2.18}$$

Finally, by using (2.18) in (2.17) and taking into account (2.12b) we deduce that $B_i{}^k{}_j = F_i{}^k{}_j$, which together with (2.16) proves the assertion of the theorem. □

Next, in order to get the local coefficients of ∇, we consider the local frame field $\{\delta/\delta x^i, \partial/\partial y^i\}$ on TM° and set

$$\nabla_{\delta/\delta x^j}\frac{\delta}{\delta x^i} = X_i{}^k{}_j\frac{\partial}{\partial y^k} + Y_i{}^k{}_j\frac{\delta}{\delta x^k}, \tag{2.19}$$

$$\nabla_{\partial/\partial y^j}\frac{\partial}{\partial y^i} = Z_i{}^k{}_j\frac{\partial}{\partial y^k} + U_i{}^k{}_j\frac{\delta}{\delta x^k}, \tag{2.20}$$

$$\nabla_{\delta/\delta x^j}\frac{\partial}{\partial y^i} = V_i{}^k{}_j\frac{\partial}{\partial y^k} + W_i{}^k{}_j\frac{\delta}{\delta x^k}. \tag{2.21}$$

Taking into account that ∇ is torsion free and using (2.4) and (2.21), we infer that

$$\nabla_{\partial/\partial y^i}\frac{\delta}{\delta x^j} = V_i{}^k{}_j\frac{\partial}{\partial y^k} + W_i{}^k{}_j\frac{\delta}{\delta x^k} - G_i{}^k{}_j\frac{\partial}{\partial y^k}. \tag{2.22}$$

Now, we replace (X,Y,Z) from (2.10) in turn by $(\delta/\delta x^j, \delta/\delta x^i, \partial/\partial y^h)$ and $(\delta/\delta x^j, \delta/\delta x^i, \delta/\delta x^h)$ and using (2.4), (2.5), (2.9), and (2.19), we obtain

$$X_i{}^k{}_j = -C_i{}^k{}_j - \frac{1}{2}R_i{}^k{}_j \qquad Y_i{}^k{}_j = F_i{}^k{}_j. \tag{2.23}$$

Similarly, we replace (X,Y,Z) from (2.10) in turn by $(\partial/\partial y^j, \partial/\partial y^i, \partial/\partial y^h)$ and $(\partial/\partial y^j, \partial/\partial y^i, \delta/\delta x^h)$ and deduce that

$$Z_i{}^k{}_j = C_i{}^k{}_j \qquad 2g_{hk}U_i{}^h{}_j = -\frac{\delta g_{ij}}{\delta x^k} + g_{hj}G_i{}^h{}_k + g_{ih}G_j{}^h{}_k. \tag{2.24}$$

As $G_i{}^k{}_j$ given by (2.6a) are the local coefficients of the Berwald connection, we obtain

$$2g_{hk}U_i{}^h{}_j = -g_{ij;k}, \tag{2.25}$$

where $g_{ij;k}$ is the h-covariant derivative of g_{ij} with respect to the Berwald connection. Next, by equation (18.24) in Matsumoto [6], we have

$$g_{ij;k} = -2C_{ijk|o} \tag{2.26}$$

and hence

$$U_i{}^k{}_j = 2C_i{}^k{}_{j|o}. \tag{2.27}$$

Finally, replace (X,Y,Z) from (2.10) in turn by $(\delta/\delta x^j, \partial/\partial y^i, \partial/\partial y^h)$ and $(\delta/\delta x^j, \partial/\partial y^i, \delta/\delta x^h)$ and using (2.4), (2.5), (2.9), and Theorem 2.1, we derive that

$$V_i{}^k{}_j = F_i{}^k{}_j, \qquad W_i{}^k{}_j = C_i{}^k{}_j + \frac{1}{2}R_{ihj}g^{hk}, \tag{2.28}$$

where we set $R_{ihj} = g_{it}R^t{}_{hj}$. Thus (2.19), (2.20), (2.21), (2.22), and the above calculations enable us to state the following theorem.

Theorem 2.2. *The Levi-Civita connection ∇ on TM° with respect to the Sasaki-Finsler metric G is locally expressed in terms of the local coefficients of the Cartan connection of $\mathbb{F}^m$ as follows:*

$$\nabla_{\delta/\delta x^j}\frac{\delta}{\delta x^i} = -\left(C_i{}^k{}_j + \frac{1}{2}R^k{}_{ij}\right)\frac{\partial}{\partial y^k} + F_i{}^k{}_j\frac{\delta}{\delta x^k}, \tag{2.29}$$

$$\nabla_{\partial/\partial y^j}\frac{\partial}{\partial y^i} = C_i{}^k{}_j\frac{\partial}{\partial y^k} + 2C_i{}^k{}_{j|o}\frac{\delta}{\delta x^k}, \tag{2.30}$$

$$\begin{aligned}\nabla_{\delta/\delta x^j}\frac{\partial}{\partial y^i} &= F_i{}^k{}_j\frac{\partial}{\partial y^k} + \left(C_i{}^k{}_j + \frac{1}{2}R_{ihj}g^{hk}\right)\frac{\delta}{\delta x^k} \\ &= \nabla_{\partial/\partial y^i}\frac{\delta}{\delta x^j} + G_i{}^k{}_j\frac{\partial}{\partial y^k}.\end{aligned} \tag{2.31}$$

3. The main result. It is well known that, on the tangent bundle TM, there exists a globally defined vector field $L = y^i(\partial/\partial y^i)$ called the *Liouville vector field*. By means of the almost product structure Q, we obtain another vector field $QL = y^i(\delta/\delta x^i)$ which we call the *horizontal Liouville vector field* of $\mathbb{F}^m$. Clearly, $\xi = \ell^i(\delta/\delta x^i)$, where $\ell^i = y^i/F$ is a unit vector field with respect to G. To state the next theorem, we recall that the angular metric of $\mathbb{F}^m$ has the local components

$$h_{ij} = g_{ij} - \ell_i\ell_j; \quad \ell_i = g_{ij}\ell^j = \frac{\partial F}{\partial y^i}. \tag{3.1}$$

Also, we recall that the Lie derivative of G with respect to ξ is given by (cf. Yano-Kon [12, page 41])

$$(L_\xi G)(X,Y) = G(\nabla_X\xi, Y) + G(\nabla_Y\xi, X) \quad \forall X,Y \in \Gamma(TTM^\circ). \tag{3.2}$$

Now we prove the following theorem.

Theorem 3.1. *The Lie derivative of G with respect to ξ satisfies the equations*

$$(L_\xi G)(vX, vY) = (L_\xi G)(hX, hY) = 0, \tag{3.3}$$

$$(L_\xi G)(hX, vY) = \frac{1}{F}(h_{ij} - R_{ioj})X^iY^j \tag{3.4}$$

for any $X, Y \in \Gamma(TTM^\circ)$, where $hX = X^i(\delta/\delta x^i)$ and $vY = Y^i(\partial/\partial y^i)$.

Proof. First, by using (2.31), (2.9) and taking into account that $N_i^k = y^jF_{i\,j}^{\,k}$, we obtain

$$\begin{aligned} G\left(\nabla_{\partial/\partial y^j}\xi, \frac{\partial}{\partial y^i}\right) &= \ell^k\left(F_{j\ k}^{\ h} - G_{j\ k}^{\ h}\right)g_{hi} \\ &= \frac{1}{F}\left(N_j^h - y^k\frac{\partial N_j^h}{\partial y^k}\right)g_{hi} \\ &= 0, \end{aligned} \tag{3.5}$$

since N_j^h are positively homogeneous of degree 1 with respect to (y^k). Next, by using (2.29) and (2.9), we deduce that

$$G\left(\nabla_{\delta/\delta x^j}\xi, \frac{\delta}{\delta x^i}\right) = g_{ki}\ell^k{}_{|j} = 0, \tag{3.6}$$

since $\ell^k{}_{|j} = 0$. Taking into account (3.2), we see that (3.5) and (3.6) yield (3.3). Finally, substituting X and Y from (3.2) by $\delta/\delta x^j$ and $\partial/\partial y^i$, respectively, and using (2.29), (2.31), and (2.9), we infer that

$$(L_\xi G)\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right) = \ell_{i\|j} - \frac{1}{F}R_{joi} = \frac{1}{F}(h_{ij} - R_{ioj}), \tag{3.7}$$

since by equations (17.30) and (17.21) in Matsumoto [6] we have $\ell_{i\|j} = (1/F)h_{ij}$ and $R_{ioj} = R_{joi}$. As (3.7) implies (3.4), the proof is complete. □

Next, for a fixed point $x \in M$ we consider the indicatrix I_x at x, which is a hypersurface of the fibre TM_x° given by the equation $F(x,y) = 1$. Then denote by IM the hypersurface of TM° consisting of indicatrices at all points of M and call it the *indicatrix bundle* over $\mathbb{F}^m$. It is easy to show that $Q\xi = \ell^i(\partial/\partial y^i)$ is the unit normal vector field with respect to the Sasaki-Finsler metric. Indeed, if the local equations of IM in TM° are

$$x^i = x^i(u^\alpha), \quad y^i = y^i(u^\alpha), \quad \alpha \in \{1,\dots,2m-1\}, \tag{3.8}$$

then, we have

$$\frac{\partial F}{\partial x^i}\frac{\partial x^i}{\partial u^\alpha} + \frac{\partial F}{\partial y^i}\frac{\partial y^i}{\partial u^\alpha} = 0. \tag{3.9}$$

As the h-covariant derivative of F vanishes, by using (2.3), we obtain

$$\left(N_i^k \frac{\partial x^i}{\partial u^\alpha} + \frac{\partial y^k}{\partial u^\alpha}\right)\ell_k = 0. \tag{3.10}$$

The natural frame field on IM is represented by

$$\frac{\partial}{\partial u^\alpha} = \frac{\partial x^i}{\partial u^\alpha}\frac{\partial}{\partial x^i} + \frac{\partial y^i}{\partial u^\alpha}\frac{\partial}{\partial y^i} = \frac{\partial x^i}{\partial u^\alpha}\frac{\delta}{\delta x^i} + \left(N_i^k \frac{\partial x^i}{\partial u^\alpha} + \frac{\partial y^k}{\partial u^\alpha}\right)\frac{\partial}{\partial y^k}. \tag{3.11}$$

Then by (3.10), we deduce

$$G\left(\frac{\partial}{\partial u^\alpha}, Q\xi\right) = \left(N_i^k \frac{\partial x^i}{\partial u^\alpha} + \frac{\partial y^k}{\partial u^\alpha}\right) y^h g_{hk} = 0. \tag{3.12}$$

Thus $Q\xi$ is orthogonal to any vector tangent to IM. The horizontal Liouville vector field is tangent to IM since $G(\xi, Q\xi) = 0$.

To state the next corollary, we recall that ξ is a Killing vector field on IM with respect to G if and only if $L_\xi G = 0$ (cf. Yano-Kon [12, page 41]). Thus, by Theorem 3.1, we may state the following corollary.

Corollary 3.2. *The unit horizontal Liouville vector field ξ is a Killing vector field on the indicatrix bundle IM if and only if*

$$h_{ij}(x,y) = R_{ioj}(x,y) \quad \forall (x,y) \in IM. \tag{3.13}$$

Now, we consider a Finsler vector field $X = X^i(\partial/\partial y^i)$ which is noncolinear to the Liouville vector field L. Then the *curvature (flag curvature)* of $\mathbb{F}^m$ for the flag spanned by $\{L, X\}$ is the function (see Equation (26.1) in Matsumoto [6] or Bao-Cheen-Shen [3])

$$K(x,y,X) = \frac{R_{ioj}X^iX^j}{F^2 h_{ij}X^iX^j}. \tag{3.14}$$

In case K is a constant we say that $\mathbb{F}^m$ is a Finsler manifold of constant curvature. The above results enable us to state a geometric characterization of Finsler manifolds of constant curvature by means of the horizontal Liouville vector field, which is the main result of this paper.

Theorem 3.3. *The Finsler manifold $\mathbb{F}^m$ is of constant curvature $K = 1$ if and only if the unit horizontal Liouville vector field is a Killing vector field on the indicatrix bundle IM.*

Proof. Suppose $K = 1$ and from (3.14) we obtain (3.13), since $F(x,y) = 1$ on IM.

Conversely, suppose ξ is a Killing vector field on IM. Then by using (3.13) in (3.14), we deduce that $K(x,y,X) = 1$ for any Finsler vector $X(x,y)$ and any $(x,y) \in IM$. Now, take a point $(x,y) \in TM^\circ \backslash IM$. Then there exists $a \in (0,\infty)\backslash\{1\}$ such that $F(x,y) = a$. As F is positive homogeneous of degree 1 with respect to y, we have $F(x,(1/a)y) = 1$. Hence $(x,(1/a)y) \in IM$ and by (3.13), we have

$$h_{ij}\left(x,\frac{1}{a}y\right) = R_{ioj}\left(x,\frac{1}{a}y\right). \tag{3.15}$$

Taking into account that h_{ij} and R_{ioj} are positively homogeneous of degrees 0 and 2, respectively, we infer that

$$R_{ioj}(x,y) = F^2(x,y)h_{ij}(x,y). \tag{3.16}$$

Thus from (3.14), we deduce $K(x,y,X) = 1$. This completes the proof. □

In the above discussions the constant curvature was taken to be $K = 1$ for "normalisation" purposes only. However the geometric characterization remains valid for any positive constant curvature. To be more precise, we give the following. For any real number $\lambda > 0$, we define the λ-indicatrix bundle $I_\lambda M$ to be:

$$I_\lambda M = \left\{(x,y) \in TM^\circ : F(x,y) = \sqrt{\frac{1}{\lambda}}\right\}. \tag{3.17}$$

Then simple modifications in the calculations given earlier will show that the unit horizontal Liouville vector field ξ is a Killing vector field on $I_\lambda M$ if and only if $h_{ij}(x,y) = R_{ioj}(x,y)$, $\forall (x,y) \in I_\lambda M$. So, as before, this can be used to prove the following theorem.

Theorem 3.4. *The Finsler manifold $\mathbb{F}^m$ is of constant positive curvature λ if and only if the unit horizontal Liouville vector field is a Killing vector field on $I_\lambda M$.*

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