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LOW REYNOLDS NUMBER STABILITY OF MHD PLANE POISEUILLE FLOW OF AN OLDROYD FLUID

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ABSTRACT. The linear stability of plane Poiseuille flow at low Reynolds number of a conducting Oldroyd fluid in the presence of a transverse magnetic field has been investigated numerically. Spectral tau method with expansions in Chebyshev polynomials is used to solve the Orr-Sommerfeld equation. It is found that viscoelastic parameters have destabilizing effect and magnetic field has a stabilizing effect in the field of flow. But no instabilities are found.

Keywords and phrases. Stability, Reynolds number, MHD, Poiseuille flow, spectral tau method.

2000 Mathematics Subject Classification. Primary 76E25.

1. Introduction. The Orr-Sommerfeld equation has been studied in details for small values of Reynolds number by Southwell and Chitty [9] for the particular case of plane Couette flow. A more general discussion was given later by Pekeris [8] who also gave detailed results for both plane Couette and plane Poiseuille flow. The problem has been considered again independently by Birikh, Gershuni, and Zhuokhovitskii [1]. Ho and Denn [2] studied low Reynolds number stability for plane Poiseuille flow by using a numerical scheme based on the shooting method. They found that at low Reynolds numbers no instabilities occur, but the numerical method led to artificial instabilities. Lee and Finlayson [3] used a similar numerical method to study both Poiseuille and Couette flow, and confirmed the absence of instabilities at low Reynolds number. In this paper, we study the linear stability of plane Poiseuille flow at small Reynolds number of a conducting Oldroyd fluid in the presence of magnetic field. The fourth-order Orr-Sommerfeld equation governing the stability analysis is solved numerically by spectral tau method with expansions in Chebyshev's polynomials following Orszag [7]. We employ Mathematica (Windows Version) in all our numerical computations to find eigenvalues.

2. Basic equations. Oldroyd model for viscoelastic fluid is described by the constitutive equations (Oldroyd [5, 6])

$$T_{ij}^{*} = -\Pi\delta_{ij} + s_{ij}, \qquad \left(1+\lambda_1\frac{d}{dt}\right)s_{ij} = 2\eta\left(1+\lambda_2\frac{d}{dt}\right)e_{ij}, \qquad e_{ij} = \frac{1}{2}(v_{i,j}+v_{j,i}), \quad (2.1)$$

where T_{ij}^{*}, s_{ij}, e_{ij}, Π, η, λ_1, λ_2 ($\lambda_1 > \lambda_2 > 0$) are, respectively, total stress tensor, deviatoric stress tensor, rate of strain tensor, pressure, coefficient of viscosity, stress relaxation time, and strain retardation time, respectively.

The stress equation of motion and the induction equation are

$$\rho(v_{i,t}+v_{i,j}v_j) = -\Pi_{,i}+s_{ij,j}, \tag{2.2}$$

$$H^*_{i,t}+v_kH^*_{i,k} = H^*_k v_{i,k}+\left(\frac{1}{\mu\sigma}\right)H^*_{i,kk}, \tag{2.3}$$

where H^* is the magnetic field intensity.

Let $2l$ be the distance between the parallel plates. The origin is taken at a point midway between the plates. The steady primary flow is taken parallel to the x-axis with y-axis normal to the plates. We apply a transverse magnetic field perpendicular to the plates in the direction of y-axis. Then the velocity field v_i and the magnetic field H_i are given by

$$(U,0,0) \qquad \text{and} \qquad (H_1^*,H_0^*,0). \tag{2.4}$$

Then the equation of motion (2.2) becomes

$$\rho(v_{i,t}+v_kv_{i,k}) = -\Pi^*_{,i}+s_{ik,k}+\mu H^*_kH^*_{i,k}, \tag{2.5}$$

where $\Pi^* = \Pi+(\mu H^{*2}/2)$.

We use the following nondimensional quantities:

$$\begin{aligned}
&\bar{x}=\frac{x}{l}, \quad \bar{y}=\frac{y}{l}, \quad \bar{v}_i=\frac{v_i}{U_0}, \quad \bar{t}=\frac{tU_0}{l}, \quad \bar{\Pi}^*=\frac{\Pi^*}{\rho U_0^2},\\
&\bar{s}_{ij}=\frac{s_{ij}}{\rho U_0^2}, \quad \bar{H}^*_i=\frac{H^*_i}{H^*_0}, \quad \alpha_1=\frac{\lambda_1U_0}{l}, \quad \alpha_2=\frac{\lambda_2U_0}{l},
\end{aligned} \tag{2.6}$$

where the central line velocity is taken as the characteristic velocity U_0.

Dropping the bar over the symbols, (2.3) and (2.5) become

$$\begin{aligned}
v_{i,t}+v_kv_{i,k} &= -\Pi^*_{,i}+s_{ik,k}+SH^*_kH^*_{i,k},\\
H^*_{i,t}+v_kH^*_{i,k} &= H^*_kv_{i,k}+\left(\frac{1}{R_m}\right)H^*_{i,kk},
\end{aligned} \tag{2.7}$$

where $R = \rho U_0l/\eta$ is the Reynolds number, $R_m = \mu\sigma lU_0$ is the magnetic Reynolds number, and $S = \mu H_0^{*2}/\rho U_0^2$ is the magnetic pressure number.

3. Solution of the basic flow. The basic flow is steady and unidirectional and is given by

$$U(y) = \frac{\cosh M-\cosh My}{\cosh M-1}, \tag{3.1}$$

where $M = \mu lH_0^*\sqrt{\sigma/\eta_0}$ is the Hartmann number and the induced magnetic field is

$$H_1(y) = R_m\frac{\sinh My-y\sinh M}{M(\cosh M-1)}. \tag{3.2}$$

4. Stability analysis. Following the usual terminology of linear stability analysis, let the disturbed flow be written as a steady basic flow plus a time dependent disturbance, assumed small,

$$v_i = U_i + u_i, \qquad \Pi^* = P^* + p^*, \qquad s_{ij} = T_{ij} + \tau_{ij}, \qquad H_i^* = H_i + h_i. \tag{4.1}$$

In our stability analysis we assume the validity of Squire's theorem, namely the 2-dimensional disturbances are more unstable than 3-dimensional ones, and therefore consider a 2-dimensional disturbance.

Now, we assume that the disturbances are periodic in the x-direction and write

$$\begin{aligned} u_j &= \hat{u}_j(y)\exp(ik(x-ct)), \qquad h_j = \hat{h}_j(y)\exp(ik(x-ct)),\\ p^* &= \hat{p}^*(y)\exp(ik(x-ct)), \qquad \tau_{ij} = \hat{\tau}_{ij}(y)\exp(ik(x-ct)), \end{aligned} \tag{4.2}$$

where k is the wave number of the disturbances, $c = c_r + ic_i$ is the complex wave speed and quantities with the carret ($\wedge$) are complex amplitudes. The motion is stable or unstable according as $c_i < 0$ or $c_i > 0$.

Writing (2.2) in component form and substituting the stress components in equations of motion and induction and then, using (3.2) we get the three stress equations.

Substituting these stress amplitudes in the two components of the equation of motion, after a tedious algebra we have,

$$\begin{aligned} &ikR\omega_1[-(U-c)v' + vU']\\ &\quad = k^2R\omega_1\hat{p}^* - [\omega_2(v''' - k^2v') + (2\omega_1' - \omega_2')v'' + (\omega_1'' + \omega_2'')v' + 2\omega_1'(\omega_1' - \omega_2')v']\\ &\qquad + k^2\omega_2'v + \left(\frac{\omega_1'}{\omega_1}\right)[\omega_2(v'' - k^2v) + 2(\omega_1' - \omega_2')v' + (\omega_1'' - \omega_2'')v + 2v\omega_1'(\omega_1' - \omega_2')]\\ &\qquad + 2(\omega_1''/\omega_1)\omega_2v' - (\omega_1''' - \omega_2''')v + 2(\omega_1''/\omega_1)(\omega_1' - \omega_2')v\\ &\qquad - SR\omega_1(ikH_1\hat{h}_2 + \hat{h}_2'' - ik\hat{h}_2H_1'), \end{aligned} \tag{4.3}$$

$$\begin{aligned} ikR\omega_1(U-c)v = &-\omega_1R\hat{p}^{*\prime} + \omega_2(v'' - k^2v) + 2(\omega_1' - \omega_2')v' + (\omega_1'' - \omega_2'')v\\ &+ 2v\omega_1'(\omega_1' - \omega_2') + SR\omega_1(ikH_1\hat{h}_2 + \hat{h}_2'), \end{aligned} \tag{4.4}$$

where ($'$) denotes the derivative with respect to y and

$$\omega_1(y) = 1 + ik\alpha_1(U-c), \qquad \omega_2(y) = 1 + ik\alpha_2(U-c), \tag{4.5}$$

$\hat{u}_1 = u$, $\hat{u}_2 = v$, also $v' = -iku$ by equation of continuity.

Eliminating $\hat{p}^*$ from (4.3) and (4.4) and neglecting the terms of second order in α_1 and α_2, we get

$$\begin{aligned} &ikR\omega_1[(U-c)(v'' - k^2v) - vU'']\\ &\quad = \omega_2v'''' - 2\omega_2k^2v'' + [\omega_2k^4 + (\omega_1'''' - \omega_2'''')]v\\ &\qquad + SR\omega_1[ik\hat{h}_2H_1'' - ikH_1\hat{h}_2'' + ik^3H_1\hat{h}_2 + k^2\hat{h}_2' - \hat{h}_2''']. \end{aligned} \tag{4.6}$$

The equation of induction becomes

$$ik\hat{h}_2(U-c) = ikH_1 v + v' + \left(\frac{1}{R_m}\right)(\hat{h}_2'' - k^2\hat{h}_2). \tag{4.7}$$

A further simplification of (4.6) and (4.7) can be effected by the fact that for most conducting fluids the ratio $R_m/R = \nu/\lambda$ is extremely small (Lock [4]). We can thus neglect all terms involving H_1 in (4.6) and (4.7) and also the term $ik\hat{h}_2(U-c)$ on the left-hand side of (4.7). Then, we eliminate $\hat{h}_2$ from these two equations and finally get the fourth-order Orr-Sommerfeld equation for the present problem as

$$\begin{aligned}\omega_2 v'''' + (M^2\omega_1 - 2\omega_2 k^2)v'' + (\omega_2 k^4 + \omega_1'''' - \omega_2'''')v \\ - ikR\omega_1\left[(U-c)(v'' - k^2 v) - vU''\right] = 0.\end{aligned} \tag{4.8}$$

The boundary conditions are

$$v = v' = 0 \quad \text{at } y = \pm 1. \tag{4.9}$$

5. Numerical computation. To solve the Orr-Sommerfeld equation (4.8) “spectral method” with expansions of velocity in terms of Chebyshev polynomials is used.

We write

$$v(y) = \sum_{n=0}^{\infty} a_n T_n(y), \qquad v''(y) = \sum_{n=0}^{\infty} a_n^{(2)} T_n(y), \qquad v''''(y) = \sum_{n=0}^{\infty} a_n^{(4)} T_n(y), \tag{5.1}$$

where

$$\begin{aligned} a_n^{(2)} &= \left(\frac{1}{q_n}\right) \sum_{p=n+2;2}^{\infty} p(p^2 - n^2)a_p, \quad n > 0, \\ a_n^{(4)} &= \left(\frac{1}{24q_n}\right) \sum_{p=n+4;2}^{\infty} p\left[p^2(p^2-4)^2 - 3n^2p^4 + 3n^4p^2 - n^2(n^2-4)^2\right]a_p, \end{aligned} \tag{5.2}$$

where $q_0 = 2$ and $q_n = 1$ for $n > 0$.

Again, we write the basic flow $U(y)$ in terms of Chebyshev polynomials as

$$U(y) = \sum_{m=0}^{\infty} b_m T_m(y) \tag{5.3}$$

and its derivatives in the same way as (5.2).

Then after some involved calculations and using the identity

$$T_n T_m = \frac{(T_{n+m} + T_{n-m})}{2}, \qquad T_k = T_{-k} \quad \text{for } k > 0, \tag{5.4}$$

we get

$$\begin{aligned}
\sum_{n=0}^{N}\Bigg\{\Bigg[&\left(\frac{A_1}{q_n}\right)\sum_{p=n+4;2}^{N} p\Big(p^2(p^2-4)^2-3n^2p^4+3n^4p^2-n^2(n^2-4)^2\Big)a_p \\
&+\left(\frac{A_2}{q_n}\right)\sum_{p=n+2;2}^{N} p(p^2-n^2)a_p+A_3a_n\Bigg]T_n \\
&+\left(\frac{A_4}{q_n}\right)\sum_{m=0;2}^{N} b_m\left(\sum_{p=n+4;2}^{N} p\Big(p^2(p^2-4)^2-3n^2p^4\right. \\
&\qquad\left.+3n^4p^2-n^2(n^2-4)^2\Big)a_p\right)\frac{(T_{n+m}+T_{n-m})}{2} \\
&+\left(\frac{A_5}{q_n}\right)\sum_{m=0;2}^{N} b_m\left(\sum_{p=n+2;2}^{N} p(p^2-n^2)a_p\right)\frac{(T_{n+m}+T_{n-m})}{2} \\
&+A_6a_n\sum_{m=0;2}^{N} b_m\frac{(T_{n+m}+T_{n-m})}{2} \\
&+A_7\sum_{m=0;2}^{N}\left(\frac{1}{q_m}\right)\sum_{p=m+4;2}^{N} p\Big(p^2(p^2-4)^2-3n^2p^4 \\
&\qquad+3n^4p^2-n^2(n^2-4)^2\Big)b_p\frac{(T_{n+m}+T_{n-m})}{2} \\
&+A_8\sum_{m=0;2}^{N}\left(\frac{1}{q_m}\right)\left(\sum_{p=m+2;2}^{N} p(p^2-n^2)b_p\right)\frac{(T_{n+m}+T_{n-m})}{2} \\
&+A_9\sum_{m=0;2}^{N}\left(\frac{g_m}{q_m}\right)\left(\sum_{p=m+2;2}^{N} p(p^2-n^2)b_p\right)\frac{(T_{n+m}+T_{n-m})}{2} \\
&-A_9\sum_{m=0;2}^{N}(k^2g_m+d_m)\frac{(T_{n+m}+T_{n-m})}{2}\Bigg\}=0,
\end{aligned} \tag{5.5}$$

where

$$UU''=\sum_{m=0}^{\infty} d_mT_m(y),\qquad U^2=\sum_{m=0}^{\infty} g_mT_m(y), \tag{5.6}$$

and

$$\begin{aligned}
A_1&=\frac{(1-ik\alpha_2c)}{24},\\
A_2&=M^2-2k^2+ikRc+k^2R\alpha_1c^2-ikM^2\alpha_1c+2ik^3\alpha_2c,\\
A_3&=k^4-k^4R\alpha_1c^2-ik^3Rc-ik^5\alpha_2c,\\
A_4&=\frac{ik\alpha_2}{24},\\
A_5&=ik(M^2\alpha_1-R-2k^2\alpha_2)-2k^2R\alpha_1c,\\
A_6&=ik^3R+ik^5\alpha_2+2k^4R\alpha_1c,
\end{aligned}$$

$$A_7 = \frac{ik(\alpha_1 - \alpha_2)}{24},$$
$$A_8 = ikR + k^2 R\alpha_1 c,$$
$$A_9 = k^2 R\alpha_1. \tag{5.7}$$

By analysing the possible combination of T_n, T_{n+m}, T_{n-m}, we can write (5.5) in a simple form as

$$\sum_{n=0}^{N} G_n(a_0, a_1, a_2, \ldots, a_N, R, k, c, \alpha_1, \alpha_2, M) T_n = 0. \tag{5.8}$$

Equating different coefficients of T_n to zero, we get a system of $N+1$ linear simultaneous equations involving the $N+1$ unknown coefficients $a_0, a_1, a_2, \ldots, a_N$, which can be written in the matrix form as

$$\begin{pmatrix} \bar{G}_0 \\ \bar{G}_1 \\ \bar{G}_2 \\ \vdots \\ \bar{G}_N \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix} = 0 \qquad \text{or} \qquad \bar{G}_{(N+1)\times(N+1)} A_{(N+1)\times 1} = 0, \tag{5.9}$$

where $\bar{G}_N$ contains R, c, α_1, α_2, k, and M.

The system of equations has a nontrivial solution if

$$\det\left(\bar{G}_{(N+1)\times(N+1)}\right) = 0. \tag{5.10}$$

Using the following properties of Chebyshev polynomials:

$$T_n(\pm 1) = (\pm 1)^n \qquad \text{and} \qquad T_n'(\pm 1) = (\pm 1)^{n-1} n^2, \tag{5.11}$$

the boundary conditions can be written as

$$\begin{aligned} \sum_{n=0}^{N} a_n = 0, \qquad & \sum_{n=0}^{N} (-1)^n a_n = 0, \\ \sum_{n=0}^{N} n^2 a_n = 0, \qquad & \sum_{n=0}^{N} n^2 (-1)^{n-1} a_n = 0. \end{aligned} \tag{5.12}$$

Following spectral method (Orszag [7]), we replace the last four rows in $\bar{G}$ by these four boundary conditions (5.12). We use computer software to evaluate the determinant $\bar{G}$ and employ the relation (5.10) to find the eigenvalue $c = c_r + ic_i$ for the problem. We take $N = 30$, that is, we have computed a 31×31 matrix so that the eigenvalues are correct up to six places of decimals (Orszag [7, page 697]).

6. Discussion. To find the eigenvalues, we apply the relation (5.10) and use the IBM software Mathematica (Windows Version) to find the eigenvalues.

TABLE 6.1. Eigenvalues with lowest imaginary part for $k = 1.0$, $\alpha_1 = 0.0$, and $\alpha_2 = 0.0$.

R	c_r	c_i
10000	1.64959	+0.02257
200	0.664695	−0.03244
100	0.901812	−0.09284
10	0.988223	−0.92691
1	0.989536	−9.32212

TABLE 6.2. Eigenvalues with lowest imaginary part for $R = 1.0$, $\alpha_1 = 0.02$, and $\alpha_2 = 0.01$.

k	c_r	c_i
1	0.989381	−10.60120
2	0.954316	−5.91207
3	0.932722	−6.0101016
4	−1.81369	−7.51125
5	−3.1986	−6.43929
7	−4.05393	−5.44365
10	−4.24607	−5.08422
20	−4.16816	−4.539
30	0.864829	−4.03067

NONMAGNETIC CASE ($M = 0$). Orszag [7] studied extensively this problem for a Newtonian fluid. We begin with the one given by Orszag [7] for Reynolds number $R = 10000$, and wave number $k = 1$. Results displayed in Table 6.1 show that this eigenvalue is highly stable at low-Reynolds number, as noted by the negative value of the imaginary part c_i of the eigenvalue and its large absolute value.

When $\alpha_1 = 0.02$ and $\alpha_2 = 0.01$ the eigenvalues with smallest imaginary parts are shown in Table 6.2 for different values of wave number k with fixed $R = 1$. From Table 6.2 we see that the magnitude of imaginary part of the eigenvalue ($|c_i|$) decreases with the increase of wave number k up to a certain value of $k = 2$ and then it increases up to a certain value of k (here $k = 4$). Thereafter the value of $|c_i|$ again decreases gradually with k. The value of c_i remain negative and is nowhere near zero. Thus the problem is stable.

When $\alpha_1 = 0.2$ and $\alpha_2 = 0.1$ for fixed $R = 1.0$, the results are tabulated in Table 6.3. Comparing these results with the values shown in Table 6.2, we see that the change in c_i becomes quite small near $k = 5$. We also observe that the value of $|c_i|$ decreases with the increase of viscoelastic parameters. This shows that the presence of viscoelastic parameters have destabilizing effect.

TABLE 6.3. Eigenvalues with lowest imaginary part for $R = 1.0$, $\alpha_1 = 0.2$, and $\alpha_2 = 0.1$.

k	c_r	c_i
1	5.91287	−4.82685
2	3.64626	−2.51653
4	2.8191	−1.89419
5	0.110322	−1.9288
7	0.179105	−1.35964
10	0.596842	−0.7665096
20	1.01726	−0.3395
30	0.194654	−0.108458

TABLE 6.4. Eigenvalues with lowest imaginary part for $R = 1.0$, $\alpha_1 = 0.2$, $\alpha_2 = 0.1$, and $M = 1.0$.

k	c_r	c_i
1	5.88929	−4.44235
2	3.65236	−2.40604
4	2.83414	−1.87385
5	2.610322	−1.72964
7	0.137379	−1.5922
10	0.609962	−0.771475
20	1.04896	−0.356781
30	0.823551	−0.148213

TABLE 6.5. Eigenvalues with lowest imaginary part for $R = 1.0$, $\alpha_1 = 0.2$, $\alpha_2 = 0.1$, and $k = 10.0$.

M	c_r	c_i
0.0	0.596842	−0.765096
1.0	0.609962	−0.771475
2.0	1.57649	−1.40931
3.0	1.52753	−1.42581

HYDROMAGNETIC CASE. The eigenvalues with lowest imaginary parts are tabulated in Table 6.4 for fixed $R = 1$, $\alpha_1 = 0.2$, $\alpha_2 = 0.1$, and $M = 1.0$ for different k. We see from Table 6.4 that the absolute value of the imaginary part of the eigenvalue $|c_i|$ decreases gradually with the increase of wave number k, but no oscillation is observed. Comparing the results with the results in Table 6.3, we see that after $k = 7$ the magnetic field has a stabilizing effect.

For different values of Hartmann number M the eigenvalues with lowest imaginary parts in magnitude are shown in Table 6.5 with fixed $R = 1.0$, $k = 10.0$, $\alpha_1 = 0.2$, and $\alpha_2 = 0.1$. We observe that $|c_i|$ increases with the increase of M. Thus the magnetic field has a stabilizing effect in the flow field. Hence, we conclude that the flow is stable.

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