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ON A GAUSS-KUZMIN TYPE PROBLEM FOR PIECEWISE FRACTIONAL LINEAR MAPS WITH EXPLICIT INVARIANT MEASURE

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ABSTRACT. A random system with complete connections associated with a piecewise fractional linear map with explicit invariant measure is defined and its ergodic behaviour is investigated. This allows us to obtain a variant of Gauss-Kuzmin type problem for the above linear map.

Keywords and phrases. Piecewise fractional linear map, random system with complete connections, ergodic behaviour, Markov operator, invariant measure, Gauss-Kuzmin type problem.

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1. Introduction. It is known that a map $T:[0,1]\to[0,1]$ is called *piecewise fractional linear* if there is a partition $D=\{0=k_0<k_1<\cdots<k_m=1\}$ such that for every $j=1,2,\dots,m$ the map is defined by

$$T(x)=\frac{A_j+B_jx}{C_j+D_jx},\quad x\in(k_{j-1},k_j),\tag{1.1}$$

where $A_jD_j-B_jC_j\neq 0$ and $T[(k_{j-1},k_j)]=(0,1)$.

According to Schweiger [9, 10] such a map which satisfies the equalities $T0=T1=0$ and $T'0<1$ is ergodic and admits an invariant measure. Its density h can be calculated explicitly and takes one of the following shapes:

$$h(x)=\frac{1}{x+a}-\frac{1}{x+\beta},\qquad h(x)=\frac{1}{(x+a)^2},\qquad h(x)=\frac{1}{x+a},\qquad h(x)=1,\tag{1.2}$$

in the special case $m=2$. But this is not true for $m\geq 3$.

In order to verify his idea, Schweiger [11] introduced a map $T:[0,1]\to[0,1]$ defined by

$$T(x)=\begin{cases}\dfrac{x}{1-(N+1)x}, & \text{if } 0\leq x\leq \dfrac{1}{N+2},\\[2mm] \dfrac{1-kx}{x}, & \text{if } \dfrac{1}{k+1}<x\leq\dfrac{1}{k} \text{ with } 1\leq k\leq N+1,\end{cases}\tag{1.3}$$

where N is a fixed positive integer.

It is obvious that

$$T^n(x)=\begin{cases}\dfrac{T^{n-1}(x)}{1-(N+1)T^{n-1}(x)}, & \text{if } 0\le T^{n-1}(x)\le \dfrac{1}{N+2},\\[2ex] \dfrac{1-kT^{n-1}(x)}{T^{n-1}(x)}, & \text{if } \dfrac{1}{k+1}<T^{n-1}(x)\le\dfrac{1}{k} \text{ with } 1\le k\le N+1\end{cases} \tag{1.4}$$

is the nth iteration of T, $n\in\{1,2,\dots\}$.

As a consequence he proved that T is a piecewise fractional linear map with $n=N+2$ branches which admits an invariant measure with density

$$h(x)=\frac{1}{x}\sum_{j=0}^{\infty}\left(\frac{1}{1+j(N+1)x}-\frac{1}{1+\big(j(N+1)+1\big)x}\right). \tag{1.5}$$

The present paper arises as an attempt to find the limit

$$\lim_{n\to\infty}\mu(T^{-n}>y)=\alpha \tag{1.6}$$

and an estimation of the error $\mu(T^{-n}>y)-\alpha$ for a pregiven nonatomic measure μ on the σ-algebra $B_{[0,1]}$ of all the Borel subsets of $[0,1]$ associated with the above explicit piecewise fractional linear map T arising in case that $m\ge 3$. That is to solve a variant of the Gauss-Kuzmin type problem for a piecewise fractional linear map with an explicit invariant measure.

Our approach is given in the context of the theory of dependence with complete connections (see [4]). For a more detailed study of the applications of dependence with complete connections to the metrical problems and other interesting aspects of number theory we refer the reader to [1, 2, 3, 5, 6, 7, 8] and others.

The paper is organized as follows. In Section 2, we define a random system with complete connections associated with the piecewise fractional linear map (1.3). In Section 3, we study the ergodic behaviour of the above random system. This gives us the possibility to obtain the asymptotic behaviour of the fact $\{T^{-n}>y\}$, $y\in[0,1]$, as $n\to\infty$, that is the associated Gauss-Kuzmin type problem.

In the next we need the following notation:

$\aleph=\{0,1,2,\dots\}$,

$\aleph^*=\{1,2,3,\dots\}$,

$B_{[0,1]}$ is the σ-algebra of all Borel subsets of $[0,1]$,

$P(X)$ is the power set of X.

2. The random system with complete connections.

Let μ be a nonatomic measure on the σ-algebra $B_{[0,1]}$. Then we may define

$$\begin{gathered}V_0(y)=\mu([0,y]),\\ V_n(y)=V_n(y,\mu)=\mu(T^n(x)<y),\quad y\in[0,1],\ x\in[0,1],\ n\in\aleph^*.\end{gathered} \tag{2.1}$$

Proposition 2.1 (the Gauss-Kuzmin type equation). *The function V_n, $n\in N$, satisfy the equation*

$$V_{n+1}(y)=V_n\left(\frac{y}{1+(N+1)y}\right)+\sum_{k=1}^{N+1}\left[1-V_n\left(\frac{1}{k+y}\right)\right],\quad y\in[0,1]. \tag{2.2}$$

Proof. Taking into account the relations (1.3) and (1.4) we obtain that, for every $y \in [0,1]$,

$$\begin{aligned} V_{n+1}(y) &= \mu(T^{n+1}(x) < y) \\ &= \mu\left(\frac{T^n(x)}{1-(N+1)T^n(x)} < y\right) + \sum_{k=1}^{N+1} \mu\left(\frac{1-kT^n(x)}{T^n(x)} < y\right) \\ &= \mu\left(T^n(x) < \frac{y}{1+(N+1)y}\right) + \sum_{k=1}^{N+1} \mu\left(T^n(x) > \frac{1}{k+y}\right) \\ &= V_n\left(\frac{y}{1+(N+1)y}\right) + \sum_{k=1}^{N+1}\left[1 - V_n\left(\frac{1}{k+y}\right)\right], \end{aligned} \tag{2.3}$$

and the proof is complete. □

Furthermore, suppose that V_0' exists and it is bounded (μ has bounded density). Then by induction we have that V_n' exists and it is bounded for any $n \in \aleph^*$. If we derive the Gauss-Kuzmin type equation (2.2) we arrive at

$$\begin{aligned} V'_{n+1}(y) = &\frac{1}{[1+(N+1)y]^2} V'_n\left(\frac{y}{1+(N+1)y}\right) \\ &+ \sum_{k=1}^{N+1} \frac{1}{(k+y)^2} V'_n\left(\frac{1}{k+y}\right), \quad y \in [0,1]. \end{aligned} \tag{2.4}$$

We denote

$$\rho(y) = \frac{1}{y} \sum_{j=0}^{\infty} \left(\frac{1}{1+j(N+1)y} - \frac{1}{1+\big(j(N+1)+1\big)y}\right) \tag{2.5}$$

and

$$f_n(y) = \frac{V'_n(y)}{\rho(y)}, \quad y \in [0,1],\ n \in \aleph. \tag{2.6}$$

Then the relation (2.4) becomes

$$\begin{aligned} f_{n+1}(y) = &\frac{\sum_{j=0}^{\infty}\left[\frac{1}{1+(j+1)(N+1)y} - \frac{1}{1+[(j+1)(N+1)+1]y}\right]}{\sum_{j=0}^{\infty}\left[\frac{1}{1+j(N+1)y} - \frac{1}{1+[j(N+1)+1]y}\right]} f_n\left(\frac{y}{1+(N+1)y}\right) \\ &+ \sum_{k=1}^{N+1} \frac{\sum_{j=0}^{\infty}\left[\frac{y}{k+y+j(N+1)} - \frac{y}{k+y+1+j(N+1)}\right]}{\sum_{j=0}^{\infty}\left[\frac{1}{1+j(N+1)y} - \frac{1}{1+[j(N+1)+1]y}\right]} f_n\left(\frac{1}{k+y}\right), \quad y \in [0,1]. \end{aligned} \tag{2.7}$$

We can now prove the following proposition.

Proposition 2.2. *The function*

$$P(y,k)=\begin{cases}\dfrac{\sum_{j=0}^{\infty}\left[\frac{1}{1+(j+1)(N+1)y}-\frac{1}{1+[(j+1)(N+1)+1]y}\right]}{\sum_{j=0}^{\infty}\left[\frac{1}{1+j(N+1)y}-\frac{1}{1+[j(N+1)+1]y}\right]}, & \text{if } k=0,\\[2ex] \sum_{k=1}^{N+1}\dfrac{\sum_{j=0}^{\infty}\left[\frac{y}{k+y+j(N+1)}-\frac{y}{k+y+1+j(N+1)}\right]}{\sum_{j=0}^{\infty}\left[\frac{1}{1+j(N+1)y}-\frac{1}{1+[j(N+1)+1]y}\right]}, & \text{if } 1\le k\le N+1,\end{cases} \tag{2.8}$$

defines a transition probability function from $([0,1],B_{[0,1]})$ *to* $(K,P(K))$*, where* $K=\{k\in\aleph \mid 0\le k\le N+1,\ N$ *fixed positive integer*$\}$.

Proof. We must prove that

$$\sum_{k=0}^{N+1}P(y,k)=1,\quad \forall y\in[0,1]. \tag{2.9}$$

Indeed since

$$\sum_{k=1}^{N+1}\sum_{j=0}^{\infty}\left[\frac{y}{k+y+j(N+1)}-\frac{y}{k+y+1+j(N+1)}\right]=\frac{y}{y+1}, \tag{2.10}$$

we have that

$$\begin{aligned}\sum_{k=0}^{N+1}P(y,k)&=P(y,0)+\sum_{k=1}^{N+1}P(y,k)\\ &=\frac{1}{\sum_{j=0}^{\infty}\left[\frac{1}{1+j(N+1)y}-\frac{1}{1+[j(N+1)+1]y}\right]}\\ &\quad\times\left(\sum_{j=0}^{\infty}\left[\frac{1}{1+(j+1)(N+1)y}-\frac{1}{1+[(j+1)(N+1)+1]y}\right]+\frac{y}{y+1}\right).\end{aligned} \tag{2.11}$$

By replacing $j+1$ with m in the second sum we obtain that

$$\begin{aligned}&\sum_{j=0}^{\infty}\left[\frac{1}{1+(j+1)(N+1)y}-\frac{1}{1+[(j+1)(N+1)+1]y}\right]+\frac{y}{y+1}\\ &\quad=\sum_{m=1}^{\infty}\left[\frac{1}{1+m(N+1)y}-\frac{1}{1+[m(N+1)+1]y}\right]+\frac{y}{y+1}\\ &\quad=\sum_{m=0}^{\infty}\left[\frac{1}{1+m(N+1)y}-\frac{1}{1+[m(N+1)+1]y}\right],\end{aligned} \tag{2.12}$$

and the proof is complete. □

Relation (2.7) and Proposition 2.2 lead to the definition of random system with complete connections (RSCC),

$$\{(Y,\mathcal{Y}),(K,\mathcal{K}),u,P\}, \tag{2.13}$$

where

$$
\begin{gathered}
Y=[0,1], \qquad \mathcal{Y}=B_{[0,1]}, \\
K=\{k\in\aleph \mid 0\le k\le N+1,\ N \text{ fixed positive integer}\}, \qquad \mathcal{K}=P(K), \\
u(y,k)=\begin{cases}\dfrac{y}{1+(N+1)y}, & \text{if } k=0,\\[2mm] \dfrac{1}{k+y}, & \text{if } 1\le k\le N+1,\end{cases} \\
P(y,k)=\begin{cases}\dfrac{\sum_{j=0}^{\infty}\left[\frac{1}{1+(j+1)(N+1)y}-\frac{1}{1+[(j+1)(N+1)+1]y}\right]}{\sum_{j=0}^{\infty}\left[\frac{1}{1+j(N+1)y}-\frac{1}{1+[j(N+1)+1]y}\right]}, & \text{if } k=0,\\[4mm] \sum_{k=1}^{N+1}\dfrac{\sum_{j=0}^{\infty}\left[\frac{y}{k+y+j(N+1)}-\frac{y}{k+y+1+j(N+1)}\right]}{\sum_{j=0}^{\infty}\left[\frac{1}{1+j(N+1)y}-\frac{1}{1+[j(N+1)+1]y}\right]}, & \text{if } 1\le k\le N+1.\end{cases}
\end{gathered}
\tag{2.14}
$$

3. The ergodic behaviour of the random system with complete connections. For any complex-valued function f defined on Y, we put

$$
|f|=\sup_{y\in Y}|f(y)|, \qquad s(f)=\sup_{y_1\ne y_2\in Y}\frac{|f(y_1)-f(y_2)|}{|y_1-y_2|}. \tag{3.1}
$$

If we denote by $L(Y)$ the set of all complex-valued measurable bounded Lipschitz functions defined on Y for which both $|f|<\infty$ and $s(f)<\infty$, then it is clear that $L(Y)$ is a Banach space under the norm $\|f\|_L=|f|+s(f)$.

Furthermore, let Q denotes the transition probability function of the Markov chain associated with RSCC (2.13). Then the Markov operator U associated with RSCC (2.13) is given by

$$
\begin{aligned}
Uf(y)&=\sum_{k\in K}P(y,k)f(u(y,k))=\int_0^1 Q(y,dy')f(y')\\
&=\frac{\sum_{j=0}^{\infty}\left[\frac{1}{1+(j+1)(N+1)y}-\frac{1}{1+[(j+1)(N+1)+1]y}\right]}{\sum_{j=0}^{\infty}\left[\frac{1}{1+j(N+1)y}-\frac{1}{1+[j(N+1)+1]y}\right]}f\left(\frac{y}{1+(N+1)y}\right)\\
&\quad+\sum_{k=1}^{N+1}\frac{\sum_{j=0}^{\infty}\left[\frac{y}{k+y+j(N+1)}-\frac{y}{k+y+1+j(N+1)}\right]}{\sum_{j=0}^{\infty}\left[\frac{1}{1+j(N+1)y}-\frac{1}{1+[j(N+1)+1]y}\right]}f\left(\frac{1}{k+y}\right), \quad y\in[0,1]
\end{aligned}
\tag{3.2}
$$

for all $f\in L(Y)$.

In order to study the ergodic behaviour of the RSCC (2.13) we prove the following.

Proposition 3.1. *RSCC (2.13) is a RSCC with contraction and its associated Markov operator given by (3.2) is regular with respect to $L(Y)$.*

Proof. We have to prove that $R_1 < \infty$ and $r_1 < 1$, where r_1 and R_1 are defined in [4]. To this end, we calculate the derivatives of u and P with respect to y. We have that

$$\begin{aligned}
&\frac{dP(y,0)}{dy}\\
&\quad= \frac{\sum_{j=0}^{\infty}\left[-\frac{(j+1)(N+1)}{[1+(j+1)(N+1)y]^2}+\frac{(j+1)(N+1)+1}{[1+((j+1)(N+1)+1)y]^2}\right]\sum_{j=0}^{\infty}\left[\frac{1}{1+j(N+1)y}-\frac{1}{1+[j(N+1)+1]y}\right]}{\left(\sum_{j=0}^{\infty}\left[\frac{1}{1+j(N+1)y}-\frac{1}{1+(j(N+1)+1)y}\right]\right)^2}\\
&\qquad-\frac{\sum_{j=0}^{\infty}\left[\frac{1}{1+(j+1)(N+1)y}-\frac{1}{1+[(j+1)(N+1)+1]y}\right]\sum_{j=0}^{\infty}\left[-\frac{j(N+1)}{[1+j(N+1)y]^2}+\frac{j(N+1)+1}{[1+(j(N+1)+1)y]^2}\right]}{\left(\sum_{j=0}^{\infty}\left[\frac{1}{1+j(N+1)y}-\frac{1}{1+(j(N+1)+1)y}\right]\right)^2},\\
&\frac{dP(y,k)}{dy}\\
&\quad= \frac{\sum_{j=0}^{\infty}\left[\frac{k+j(N+1)}{[k+y+j(N+1)]^2}-\frac{k+1+j(N+1)}{[k+y+1+j(N+1)]^2}\right]\sum_{j=0}^{\infty}\left[\frac{1}{1+j(N+1)y}-\frac{1}{1+[j(N+1)+1]y}\right]}{\left(\sum_{j=0}^{\infty}\left[\frac{1}{1+j(N+1)y}-\frac{1}{1+(j(N+1)+1)y}\right]\right)^2}\\
&\qquad-\frac{\sum_{j=0}^{\infty}\left[\frac{y}{k+y+j(N+1)}-\frac{y}{k+y+1+j(N+1)}\right]\sum_{j=0}^{\infty}\left[-\frac{j(N+1)}{[1+j(N+1)y]^2}+\frac{j(N+1)+1}{[1+(j(N+1)+1)y]^2}\right]}{\left(\sum_{j=0}^{\infty}\left[\frac{1}{1+j(N+1)y}-\frac{1}{1+(j(N+1)+1)y}\right]\right)^2},
\end{aligned}\tag{3.3}$$

where $1 \le k \le N+1$ for every $y \in [0,1]$.

Also

$$\frac{du(y,k)}{dy} = \begin{cases} \dfrac{1}{[1+(N+1)y]^2}, & \text{if } k=0,\\[2mm] -\dfrac{1}{(k+y)^2}, & \text{if } 1 \le k \le N+1 \end{cases}\tag{3.4}$$

for every $y \in [0,1]$.

Therefore

$$\begin{aligned}
&\sup_{y\in Y}\left|\frac{dP(y,k)}{dy}\right| < \infty, \quad k \in K,\\
&\sup_{y\in Y}\left|\frac{du(y,0)}{dy}\right| = 1, \quad \text{if } k=0,\\
&\sup_{y\in Y}\left|\frac{du(y,k)}{dy}\right| \le \frac{1}{k^2}, \quad \text{if } 1 \le k \le N+1.
\end{aligned}\tag{3.5}$$

It follows (cf. [4, pages 177–178]) that $R_1 < \infty$ and $r_1 < 1$, that is RSCC (2.13) is a RSCC with contraction. □

Now, in order to prove the regularity of the associated Markov operator U defined by (3.2) with respect to $L(Y)$ we have, according to Theorem 3.2.13 in [4], to find an element y^* in $[0,1]$ such that

$$\lim_{n\to\infty}\left|\sigma_n(y)-y^*\right| = 0 \tag{3.6}$$

for any $y \in Y$. Here $\sigma_n(y)$ denotes the support of $Q^n(y,\cdot)$, where Q^n is the kernel of U^n.

Let $y \in Y$. We define the iterative relation

$$y_{n+1} = \frac{1}{y_n + 1}, \quad n \in \aleph, \text{ with } y_0 = y. \tag{3.7}$$

It is clear that $y_n \in (0,1)$, $n \in \aleph^*$. Letting $n \to \infty$ in (3.7) we have

$$0 < y^* = \lim_{n\to\infty} y_n = \frac{\sqrt{5}-1}{2} < 1. \tag{3.8}$$

Now, let A be an open set in $[0,1]$ such that $y_{n+1} \in A$. Then

$$Q(y_n, A) = \sum_{\{k\in K,\ u(y_n,k)\in A\}} P(y_n, k) \tag{3.9}$$

is strictly positive. So $y_{n+1} \in \sigma_1(y_n)$ and using [4, Lemma 3.2.14] we obtain by induction that $y_n \in \sigma_n(y)$, for any $n \in \aleph^*$. Since

$$|\sigma_n(y) - y^*| = \inf_{y' \in \sigma_n(y)} |y^* - y'| \le |y^* - y_n| \xrightarrow[n\to\infty]{} 0, \tag{3.10}$$

we obtain that U is regular, that is the desired result.

By virtue of Proposition 3.1, it follows from [4, Theorem 3.4.5] that RSCC (2.13) is uniformly ergodic. Moreover, Theorem 3.1.24 [4] implies that $Q^n(\cdot,\cdot)$ converges uniformly to a unique probability measure γ on $\mathcal{Y}$, which is stationary for the kernel Q, that is

$$\gamma(B) = \int_0^1 Q(y,B)\gamma(dy), \tag{3.11}$$

where

$$Q(y,B) = \sum_{k\in B_y} P(y,k), \tag{3.12}$$

with

$$\begin{aligned} B_y &= \{k \in K \mid u(y,k) \in B\} \\ &= \begin{cases} \{k = 0 \mid u(y,k) \in B\} \equiv B_y^{(1)}, \\ \{k \in K\backslash\{0\} \mid u(y,k) \in B\} \equiv B_y^{(2)}, \end{cases} \quad \forall B \in \mathcal{Y},\ y \in [0,1]. \end{aligned} \tag{3.13}$$

Furthermore, there exist two positive constants $q < 1$ and c such that

$$\|U^n f - U^\infty f\|_L \le c q^n \|f\|_L \tag{3.14}$$

for all $n \in \aleph^*$, $f \in L(Y)$, where

$$U^\infty f = \int_0^1 f(y)\gamma(dy). \tag{3.15}$$

In general, the form of γ cannot be determined but this is possible in our case as we prove in the following proposition.

PROPOSITION 3.2. *The probability measure γ has the density*

$$\rho(y)=\frac{1}{y}\sum_{j=0}^{\infty}\left[\frac{1}{1+j(N+1)y}-\frac{1}{1+(j(N+1)+1)y}\right],\quad y\in[0,1].\tag{3.16}$$

PROOF. By virtue of uniqueness of γ we have to show that it satisfies (3.11). Since the intervals $[0,u]$, $0<u\le 1/(N+2)$ and $(v,1/k]$, $1/(k+1)\le v<1/k$ with $k=1,2,\dots,N+1$ generate $B_{[0,1]}$ it is sufficient to verify (3.11) only for $B=[0,u]$ and $B=(v,1/k]$.

Suppose that $B=[0,u]$ with $0<u\le 1/(N+2)$. Then for $y\in[0,1]$ we have

$$B_y^{(1)}=\left\{k=0\;\Big|\;\frac{y}{1+(N+1)y}\in[0,u]\right\}=\left\{k=0\;\Big|\;0\le y\le\frac{u}{1-(N+1)u}\right\},\tag{3.17}$$

$$B_y^{(2)}=\left\{1\le k\le N+1\;\Big|\;\frac{1}{k+y}\in[0,u]\right\}=\{1\le k\le N+1\mid k\ge u^{-1}-y\}=\varnothing.\tag{3.18}$$

So by (3.12) we have that

$$Q(y,[0,u])=P(y,0)=\frac{\sum_{j=0}^{\infty}\left[\frac{1}{1+(j+1)(N+1)y}-\frac{1}{1+[(j+1)(N+1)+1]y}\right]}{\sum_{j=0}^{\infty}\left[\frac{1}{1+j(N+1)y}-\frac{1}{1+[j(N+1)+1]y}\right]}.\tag{3.19}$$

Consequently, we obtain that

$$\begin{aligned}
&\int_0^1 Q(y,[0,u])\rho(y)dy\\
&\quad=\int_0^{u/(1-(N+1)u)}P(y,0)\rho(y)dy\\
&\quad=\int_0^{u/(1-(N+1)u)}\sum_{j=0}^{\infty}\left[\frac{1}{y[1+(j+1)(N+1)y]}-\frac{1}{y[1+((j+1)(N+1)+1)y]}\right]dy\\
&\quad=\sum_{j=0}^{\infty}\left[-\log\left(1+(j+1)(N+1)\frac{u}{1-(N+1)u}\right)\right.\\
&\qquad\qquad\left.+\log\left(1+((j+1)(N+1)+1)\frac{u}{1-(N+1)u}\right)\right]\\
&\quad=\sum_{j=0}^{\infty}\left[\log\frac{1+(j(N+1)+1)u}{1+j(N+1)u}\right]=\rho([0,u]).
\end{aligned}\tag{3.20}$$

Hence (3.11) is verified for $B=[0,u]$, $0<u\le 1/(N+2)$.

The case $B=(v,1/k]$, $1/(k+1)\le v<1/k$, $k=1,2,\dots,N+1$ can be treated in a similar manner. Analogously for $y\in[0,1]$ we have that

$$\begin{aligned}
B_y^{(1)}&=\left\{k=0\;\Big|\;\frac{y}{1+(N+1)y}\in\left(v,\frac{1}{k}\right]\right\}=\varnothing,\\
B_y^{(2)}&=\left\{1\le k\le N+1\;\Big|\;\frac{1}{k+y}\in\left(v,\frac{1}{k}\right]\right\}=\left\{1\le k\le N+1\mid 0\le y<\frac{1-kv}{v}\right\}.
\end{aligned}\tag{3.21}$$

By using (3.12) we obtain that

$$\begin{aligned} Q\Big(y,\Big(v,\frac{1}{k}\Big]\Big) &= \sum_{k\in B_y} P(y,k) = \sum_{k=1}^{N+1} P(y,k) \\ &= \sum_{k=1}^{N+1} \frac{\sum_{j=0}^{\infty}\Big[\frac{y}{k+y+j(N+1)} - \frac{y}{k+y+1+j(N+1)}\Big]}{\sum_{j=0}^{\infty}\Big[\frac{1}{1+j(N+1)y} - \frac{1}{1+[j(N+1)+1]y}\Big]}. \end{aligned} \tag{3.22}$$

As a consequence we have

$$\begin{aligned} &\int_0^{(1-kv)/v} Q\Big(y,\Big(v,\frac{1}{k}\Big]\Big)\rho(y)dy \\ &\quad= \sum_{k=1}^{N+1}\int_0^{(1-kv)/v} P(y,k)\rho(y)dy \\ &\quad= \sum_{k=1}^{N+1}\int_0^{(1-kv)/v} \sum_{j=0}^{\infty}\Big[\frac{1}{k+y+j(N+1)} - \frac{1}{k+y+1+j(N+1)}\Big]dy \\ &\quad= \sum_{k=1}^{N+1}\sum_{j=0}^{\infty}\Big[\log\Big(k+\frac{1-kv}{v}+j(N+1)\Big) - \log\big(k+j(N+1)\big) \\ &\qquad\qquad + \log\Big(k+\frac{1-kv}{v}+1+j(N+1)\Big) - \log\big(k+1+j(N+1)\big)\Big] \\ &\quad= \sum_{k=1}^{N+1}\sum_{j=0}^{\infty}\Big[\log\frac{1+j(N+1)v}{1+(j(N+1)+1)v} + \log\frac{k+1+j(N+1)}{k+j(N+1)}\Big] \\ &\quad= \sum_{k=1}^{N+1}\rho\Big(\Big(v,\frac{1}{k}\Big]\Big). \end{aligned} \tag{3.23}$$

Hence (3.11) is also verified for $B=(v,1/k]$, $1/(k+1)\le v<1/k$ with $k=1,2,\ldots,N+1$ and the proof is complete. □

4. A version of the Gauss-Kuzmin type theorem. Now, we are able to determine the limit of $\mu(T^{-n}>y)$ as $n\to\infty$ and give the rate of this convergence.

Proposition 4.1. *If the density V_0' of μ is a Riemann integrable function, then*

$$\lim_{n\to\infty}\mu(T^{-n}(x)>y) = \sum_{j=0}^{\infty}\Big[\log\frac{y+1+j(N+1)}{y+j(N+1)}\Big], \quad y\ge 1,\ x\in[0,1]. \tag{4.1}$$

If the density V_0' of μ is an element of $L(Y)$, then there exist two positive constants c and $q<1$ such that

$$\lim_{n\to\infty}\mu(T^{-n}(x)>y) = (1+\theta q^n)\sum_{j=0}^{\infty}\Big[\log\frac{y+1+j(N+1)}{y+j(N+1)}\Big], \quad x\in[0,1] \tag{4.2}$$

for all $y\ge 1$, $n\in\aleph^*$, where $\theta=\theta(\mu,n,y)$ with $|\theta|\le c$.

Proof. Let $V_0' \in L(Y)$. Then $f_0 \in L(Y)$ and by using (3.15) we have

$$U^{\infty} f_0 = \int_0^1 f_0(y)\gamma(dy) = \int_0^1 V_0'(y)\,dy = 1. \tag{4.3}$$

According to relation (3.14) there exist two positive constants c and $q < 1$ such that

$$U^n f_0 = U^{\infty} f_0 + G^n f_0, \quad n \in N^*, \tag{4.4}$$

with $\|G^n f_0\|_L \le c q^n$.

If we consider the Banach space $C([0,1])$ of all real continuous functions defined on $[0,1]$ with the norm $|\cdot| = \sup|\cdot|$, then since $L([0,1])$ is a dense subset of $C([0,1])$ we have

$$\lim_{n\to\infty} \left| G^n f_0 \right| = 0 \quad \forall f_0 \in C([0,1]). \tag{4.5}$$

This means that (4.5) is valid for any measurable function f_0 which is γ-almost surely continuous, that is, for any Riemann integrable function f_0. Consequently, we obtain

$$\begin{aligned} \lim_{n\to\infty} \mu\left(T^{-n}(x) > y\right) &= \lim_{n\to\infty} V_n\left(\frac{1}{y}\right) = \lim_{n\to\infty} \int_0^{1/y} U^n f_0(w)\rho(w)\,dw \\ &= \int_0^{1/y} \rho(w)\,dw = \sum_{j=0}^{\infty} \left[\log \frac{y+1+j(N+1)}{y+j(N+1)}\right], \end{aligned} \tag{4.6}$$

that is the solution of the associated Gauss-Kuzmin type theorem. □

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