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SUBORDINATION BY CONVEX FUNCTIONS

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ABSTRACT. Let $K(\alpha)$, $0 \le \alpha < 1$, denote the class of functions $g(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are regular and univalently convex of order α in the unit disc U. Pursuing the problem initiated by Robinson in the present paper, among other things, we prove that if f is regular in U, $f(0) = 0$, and $f(z) + zf'(z) \prec g(z) + zg'(z)$ in U, then (i) $f(z) \prec g(z)$ at least in $|z| < r_0$, $r_0 = \sqrt{5}/3 = 0.745\ldots$ if $f \in K$; and (ii) $f(z) \prec g(z)$ at least in $|z| < r_1$, $r_1((51 - 24\sqrt{2})/23)^{1/2} = 0.8612\ldots$ if $g \in K(1/2)$.

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1. Introduction. Let S denote the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that are regular and univalent in the unit disc $U = \{z/|z| < 1\}$. For a given α, $0 \le \alpha < 1$, denote by $K(\alpha)$ the subclass of S consisting of functions f which satisfy the condition

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad z \in U. \tag{1.1}$$

$K(\alpha)$ is called the class of convex functions of order α and $K = K(0)$ is the class of convex functions.

Suppose that f and g are regular in $|z| < \rho$ and $f(0) = g(0)$. In addition, suppose that g is also univalent in $|z| < \rho$. We say that f is subordinate to g in $|z| < \rho$ (in symbols, $f(z) \prec g(z)$ in $|z| < \rho$) if $f(|z| < \rho) \subset g(|z| < \rho)$.

In 1947, Robinson [2] proved that if $g(z) + zg'(z)$ is in S and $f(z) + zf'(z) \prec g(z) + zg'(z)$ in $|z| < 1$, then $f(z) \prec g(z)$ at least in $|z| < r_0 = 1/5$. Subsequently, Singh and Singh [4] increased the constant r_0 to $2 - \sqrt{3} = 0.268\ldots$. Miller, Mocanu, and Read [1] further increased the constant to $4 - \sqrt{13} = 0.3944\ldots$.

Here, we consider the problem of Robinson when $g \in K$ and $K(1/2)$, respectively. (It is easy to see that $g(z) + zg'(z)$ is close-to-convex and hence univalent in $|z| < 1$ when $g \in K$.) We remark that our method works even when $g \in K(\alpha)$. However, calculations in this general case become so cumbersome that the result obtained does not commensurate with the input labour. We, therefore, confine ourselves to the particular cases $\alpha = 0$ and $\alpha = 1/2$.

2. Preliminaries. We need the following results.

LEMMA 2.1. *Suppose that f and g are regular in U, $f(0) = g(0)$, and $g'(0) \ne 0$. Suppose further that*

$$\operatorname{Re}\left(1 + \frac{zg''(z)}{g'(z)}\right) > -\frac{1}{2}, \quad z \in U. \tag{2.1}$$

Then if $f(z) \prec g(z)$ *in* U, *we have*

$$\frac{1}{z}\int_0^z f(t)\,dt \prec \frac{1}{z}\int_0^z g(t)\,dt, \quad z \in U. \tag{2.2}$$

We observe that (2.1) implies that g is close-to-convex and hence univalent in U and that the right-hand side function in (2.2) is convex in U [3]. Lemma 2.1 is due to Miller, Mocanu, and Reade [1].

The underlying idea of the following result is essentially due to Zomorvič [6] (also, see [5]).

Lemma 2.2. *Let* P *be regular in* U, $P(0) = 1$, *and* $\operatorname{Re}P(z) > 0$ *in* U. *Let* μ *and* λ *be fixed real numbers,* $-\infty < \mu < \infty$, $\lambda \geq 0$, *and* $|z| = r < 1$. *Then*

$$\operatorname{Re}\left[\mu P(z) + \frac{zP'(z)}{P(z)+\lambda}\right] \geq \begin{cases} -\left(\sqrt{\lambda(\mu+1)} - \sqrt{(a+\lambda)}\right)^2, & \text{if } \dfrac{\lambda(a+\lambda)}{(a-\rho+\lambda)^2} \geq \mu+1 \geq \dfrac{\lambda(a+\lambda)}{(a+\rho+\lambda)^2}, \\ (a-\rho)\left(\mu - \dfrac{\rho}{a-\rho+\lambda}\right), & \text{if } \mu+1 > \dfrac{\lambda(a+\lambda)}{(a-\rho+\lambda)^2}, \\ (a+\rho)\left(\mu + \dfrac{\rho}{a+\rho+\lambda}\right), & \text{if } \mu+1 < \dfrac{\lambda(a+\lambda)}{(a+\rho+\lambda)^2}, \end{cases} \tag{2.3}$$

where $a = (1+r^2)/(1-r^2)$ *and* $\rho = 2r/(1-r^2)$.

Proof. Making use of the inequality (2.3) (see [5])

$$\left| zP'(z) - \frac{P^2(z)-1}{2} \right| \leq \frac{\rho^2 - \rho_0^2}{2}, \tag{2.4}$$

where $|P(z) - a| = \rho_0 \leq \rho$, we get

$$\operatorname{Re}\left[\mu P(z) + \frac{zP'(z)}{P(z)+\lambda}\right] \geq \operatorname{Re}\left[\mu P(z) + \frac{P(z)-\lambda}{2} + \frac{(\lambda^2-1)(\overline{P(z)}+\lambda)}{2|P(z)+\lambda|^2}\right] - \frac{\rho^2-\rho_0^2}{2|P(z)+\lambda|}. \tag{2.5}$$

Taking $P(z) = a+\xi+i\eta$ and $R_1^2 = (a+\xi+\lambda)^2+\eta^2$, we get

$$\begin{aligned} \operatorname{Re}\left[\mu P(z) + \frac{zP'(z)}{P(z)+\lambda}\right] &\geq \mu(a+\xi) + \frac{a+\xi-\lambda}{2} + \frac{(\lambda^2-1)(a+\xi+\lambda)}{2R_1^2} - \frac{\rho^2-\xi^2-\eta^2}{2R_1} \\ &= S(\xi,\eta). \end{aligned} \tag{2.6}$$

Now it is easy to see that $\partial S(\xi,\eta)/\partial\eta = 0$ and $\partial^2 S(\xi,\eta)/\partial\eta^2 > 0$ at $\eta = 0$. Therefore,

$$\begin{aligned} \min_\eta S(\xi,\eta) &= S(\xi,0) \\ &= \mu(a+\xi)\frac{a+\xi-\lambda}{2} + \frac{\lambda^2-1}{2(a+\xi+\lambda)} - \frac{\rho^2-\xi^2}{2(a+\xi+\lambda)} \\ &= (\mu+1)R + \frac{\lambda(a+\lambda)}{R} - (\mu+2)\lambda - a \\ &= L(R), \end{aligned} \tag{2.7}$$

where $R = a+\xi+\lambda$. Now, using the fact that $|R(z)-a| < \rho$, we obtain the inequality

$$a-\rho+\lambda \le R \le a+\rho+\lambda. \tag{2.8}$$

It is observed that at $R = R_0 = (\lambda(a+\lambda)/(\mu+1))^{1/2}$, $\partial L(R)/\partial R = 0$ and $\partial^2 L(R)/\partial R^2 > 0$. Thus, $R = R_0$ gives the minimum value of $L(R)$ provided R_0 lies in the range of R. In view of (2.8), this is the case if the inequality

$$\frac{\lambda(a+\lambda)}{(a-\rho+\lambda)^2} \ge \mu+1 \ge \frac{\lambda(a+\lambda)}{(a+\rho+\lambda)^2} \tag{2.9}$$

is satisfied. Thus, if (2.9) holds, we have

$$\min_R L(R) = L(R_0) = -\left(\sqrt{\lambda(\mu+1)}-\sqrt{\lambda+a}\right)^2. \tag{2.10}$$

Also, it is easy to check that when $\mu+1 > \lambda(a+\lambda)/(a-\rho+\lambda)^2$, $L(R)$ is an increasing function of R. Therefore, in this case,

$$\min_R L(R) = L(a-\rho+\lambda) = (a-\rho)\left(\mu-\frac{\rho}{a-\rho+\lambda}\right). \tag{2.11}$$

On the other hand, when $\mu+1 < \lambda(a+\lambda)/(a+\rho+\lambda)^2$, $L(R)$ is a decreasing function of R. Therefore, in this case,

$$\min_R L(R) = L(a+\rho+\lambda) = (a+\rho)\left(\mu+\frac{\rho}{a+\rho+\lambda}\right). \tag{2.12}$$

This completes the proof of Lemma 2.2. □

3. Theorems and their proofs

Theorem 3.1. *Let f be regular in U with $f(0) = 0$ and let $g \in K$. Suppose that*

$$f(z)+zf'(z) \prec g(z)+zg'(z) \quad \text{in } U. \tag{3.1}$$

Then $f(z) \prec g(z)$ at least in $|z| < r_0$, where $r_0 = \sqrt{5}/3 = 0.745\ldots$.

Proof. Let us take

$$h(z) = g(z)+zg'(z). \tag{3.2}$$

Since $g \in K$, we can put

$$1+\frac{zg''(z)}{g'(z)} = P(z), \tag{3.3}$$

where $P(z)$ is regular in U, $P(0) = 1$, and $\operatorname{Re} P(z) > 0$ in U. Now, from (3.2) and (3.3), we get

$$1+\frac{zh''(z)}{h'(z)} = P(z)+\frac{zP'(z)}{P(z)+1}. \tag{3.4}$$

Taking $\mu = \lambda = 1$ in Lemma 2.2, we easily obtain

$$\operatorname{Re}\left(1+\frac{zh''(z)}{h'(z)}\right) \geq \begin{cases} \dfrac{1-2r}{1+r}, & \text{if } 0 \leq r < \dfrac{3}{5}, \\ -2\left[1-\dfrac{a}{\sqrt{1-r^2}}\right]^2, & \text{if } \dfrac{3}{5} \leq r < 1, \end{cases} \tag{3.5}$$

where $|z| = r < 1$. Now, it is easy to verify that for $0 \leq r < 3/5$, $\operatorname{Re}(1+zh''(z)/h'(z)) > -1/2$ and for $3/5 \leq r < 1$, $\operatorname{Re}(1+zh''(z)/h'(z)) > -1/2$ whenever $9r^4+22r^2-15<0$ or whenever $r < r_0$, where $r_0 = \sqrt{5}/3$ is the smallest positive root of $9r^4+22r^2-15=0$. The assertion of our theorem now follows from Lemma 2.1. □

Theorem 3.2. *Let f be regular in U with $f(0)=0$ and let $g \in K(1/2)$. Suppose that*

$$f(z)+zf'(z) \prec g(z)+zg'(z) \quad \text{in } U. \tag{3.6}$$

Then

$$f(z) \prec g(z) \tag{3.7}$$

at least in $|z| < r_1$, where $r_1 = ((51-24\sqrt{2})/23)^{1/2} = 0.8612\ldots$.

Proof. Let us put

$$h(z) = g(z)+zg'(z). \tag{3.8}$$

Since $g \in K(1/2)$, we can write

$$1+\frac{zg''(z)}{g'(z)} = \frac{P(z)+1}{2}, \tag{3.9}$$

where $P(z)$ is regular in U, $P(0)=1$, and $\operatorname{Re}P(z) > 0$ in U. From (3.8) and (3.9), we obtain

$$1+\frac{zh''(z)}{h'(z)} = \frac{1}{2}+\frac{P(z)}{2}+\frac{zP'(z)}{P(z)+3}. \tag{3.10}$$

Using Lemma 2.2 (with $\mu = 1/2$ and $\lambda = 3$), we obtain, after some calculations,

$$\operatorname{Re}\left[1+\frac{zh''(z)}{h'(z)}\right] \geq \begin{cases} \dfrac{2}{(1+r)(2+r)}, & \text{if } 0 \leq r < \dfrac{-1+\sqrt{5}}{2}, \\ 6\left[\dfrac{2-r^2}{1-r^2}\right]^{1/2} - 2\left(\dfrac{4-3r^2}{1-r^2}\right), & \text{if } \dfrac{-1+\sqrt{5}}{2} \leq r < 1, \end{cases} \tag{3.11}$$

where $|z| = r < 1$.

Now, we can easily check that for $0 \leq r < (-1+\sqrt{5})/2$, $\operatorname{Re}(1+zh''(z)/h'(z)) > -1/2$ and for $(-1+\sqrt{5})/2 \leq r < 1$, $\operatorname{Re}(1+zh'(z)/h'(z)) > -1/2$ whenever $23r^4-102r^2+63>0$ or whenever $r < r_1$, where $r_1 = ((51-24\sqrt{2})/23)^{1/2}$ is the smallest positive root of $23r^4-102r^2+63=0$. The desired result now follows from Lemma 2.1. □

In the following theorem, we take for g some distinguished members of K.

Theorem 3.3. *Let f be regular in U with $f(0)=0$ and let $f(z)+zf'(z) \prec g(z)+zg'(z)$ in U. Then*

(a) *$f(z) \prec g(z)$ in U if $g(z) = z/(1-z)$;*

(b) *$f(z) \prec g(z)$ at least in $|z| < \rho_1 = ((28 - 8\sqrt{7})/7)^{1/2} = 0.98\ldots$ if $g(z) = -\log(1-z)$;*
(c) *$f(z) \prec g(z)$ in U if $g(z) = z + \lambda z^2$, $|\lambda| \le 1/5$;*
(d) *$f(z) \prec g(z)$ at least in $|z| < \rho_2 = (9 - \sqrt{33})/4 = 0.8138\ldots$ if $g(z) = e^z - 1$.*

We observe that the functions g defined in (a), (b), (c), and (d) belong to K, $K(1/2)$, $K(1/3)$, and K, respectively.

Proof. We omit the proofs of parts (a), (c), and (d) and proceed to prove part (b). Let $h(z) = g(z) + zg'(z)$, where $g(z) = -\log(1-z)$. Then $h(0) = 0$ and $h'(0) \neq 0$. A simple computation shows that the condition

$$\operatorname{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2} \tag{3.12}$$

is equivalent to

$$\operatorname{Re}\left[\frac{2}{(1-z)(2-z)} + \frac{1}{2}\right] > 0. \tag{3.13}$$

If we let $z = re^{i\theta}$, $0 \le r < 1$ and $0 \le \theta \le 2\pi$, then condition (3.13) takes the form

$$\varphi(x) = 16r^2x^2 - 6r(4+r^2)x + r^4 + r^2 + 12 > 0, \tag{3.14}$$

where $x = \cos\theta$, $0 \le \theta \le 2\pi$. For $r = 0$, (3.14) is obviously satisfied. We, therefore, let $r \neq 0$. Now, it can be readily verified that at $x = x_0 = (12 + 3r^2)/16r$, we have $\varphi'(x) = 0$ and $\varphi''(x) > 0$.

Thus, $x = x_0$ gives the minimum value of $\varphi(x)$ provided $-1 \le x_0 \le 1$. This is true if $r \ge \rho_0 = (8 - \sqrt{28})/3 = 0.9028\ldots$. Therefore, for $r \in [\rho_0, 1]$,

$$\min_{x \in [-1,1]} \varphi(x) = \varphi(x_0) = \frac{7r^4 - 56r^2 + 48}{16}. \tag{3.15}$$

Hence, in this case, (3.14) is satisfied if $7r^4 - 56r^2 + 48 > 0$, i.e., if $r < \rho_1 = ((28 - 8\sqrt{7})/7)^{1/2} = 0.98\ldots$. Also, for $r \in [0, \rho_0)$, we can easily verify that $\varphi(x)$ is a decreasing function of x. Hence, in this case,

$$\begin{aligned} \min_{x \in [-1,1]} \varphi(x) &= \varphi(1) = {}^4 - 6^3 + 17^2 - 24 + 12 \\ &= (1-r)(2-r)(r^2 - 3r + 6) > 0. \end{aligned} \tag{3.16}$$

Therefore, we conclude that for $0 \le r < \rho_1$,

$$\operatorname{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}. \tag{3.17}$$

Conclusion (b) now follows in view of Lemma 2.1. □

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