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ON THE PROJECTIONS OF LAPLACIANS UNDER RIEMANNIAN SUBMERSIONS

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Abstract. We give a condition on Riemannian submersions from a Riemannian manifold M to a Riemannian manifold N which will ensure that it induces a differential operator on N from the Laplace-Beltrami operator on M. Equivalently, this condition ensures that a Riemannian submersion maps Brownian motion to a diffusion.

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1. Introduction. Suppose that M, N are, respectively, m- and n-dimensional Riemannian manifolds and that $m > n$. Both M and N will then carry Laplace-Beltrami operators $\triangle_M$ and $\triangle_N$, respectively, determined by the Riemannian metrics.

Let the mapping $\pi : M \to N$ such that $\pi(\sigma_m) = \sigma_n$ be a Riemannian submersion. Normally, the Laplace-Beltrami operator $\triangle_M$ will not induce a differential operator on N under the submersion π because $\triangle_M$ may depend not only on $\pi(\sigma_m)$ but also on σ_m. Equivalently, a Brownian motion on M will not normally be mapped by π to a diffusion on N because it may happen that our prediction of $\sigma_n(t+u)$ $(u > 0)$ will be improved if we know where $\sigma_m(t)$ lies in $\pi^{-1}(\sigma_n(t))$, and we can expect to get information about $\sigma_n(t)$ from the past history $\{\sigma_n(t-v) : 0 \le v < t\}$ of the submersed process. However, once we know that there is a differential operator $\mathscr{L}$ on N that satisfies the relation

$$(\mathscr{L}\phi)\circ\pi = \triangle_M(\phi\circ\pi), \tag{1.1}$$

we can find several equivalent expressions for $\mathscr{L}$ in terms of the volume, the second fundamental form, and the mean curvature of the fibres, respectively, which will be listed here.

(a) If the fibres are compact, let $v(\sigma_n)$ be the $(m-n)$-dimensional volume of the fibre $\pi^{-1}(\sigma_n)$ and V the vector field $\operatorname{grad}(\log v)$. Carne's formula (cf. [3]) then tells us that

$$\mathscr{L}\phi = \triangle_N\phi + V\phi = \triangle_N\phi + \langle V, \operatorname{grad}\phi\rangle. \tag{1.2}$$

(b) Recall that $\triangle_M$ can be written in terms of any given orthonormal vector fields $X_1,\dots,X_m$ on M as

$$\triangle_M = \sum_{i=1}^{m}\{X_iX_i - \nabla_{X_i}X_i\}, \tag{1.3}$$

the operator ∇ here being the Levi-Civita connection. Therefore, we choose $Y_1,\dots,Y_n$ to be orthonormal vector fields in a neighborhood of $\sigma_n \in N$, $X_1,\dots,X_n$ the unique horizontal lifts of $Y_1,\dots,Y_n$ to a neighborhood of $\sigma_m \in \pi^{-1}(\sigma_n)$ (so that $X_1,\dots,X_n$ are orthonormal vector fields on the π-related horizontal subspace of $\mathcal{T}(M)$) and then supplement the latter by $m-n$ orthonormal vertical vector fields $X_{n+1},\dots,X_m$ in the same neighborhood. $\triangle_M$ at σ_m can thus be written as

$$\triangle_M = \sum_{i=1}^{n} \{X_iX_i - \nabla_{X_i}X_i\} + \sum_{i=n+1}^{m} \{X_iX_i - \nabla_{X_i}X_i\}. \tag{1.4}$$

However, for any smooth function $\phi : N \to \mathbb{R}$, the composed function $\phi \circ \pi : M \to \mathbb{R}$ will be constant along each fibre $\pi^{-1}(\sigma_n)$, and hence

$$X_i(\phi\circ\pi) = \begin{cases} (Y_i\phi)\circ\pi, & 1 \le i \le n, \\ 0, & n+1 \le i \le m. \end{cases} \tag{1.5}$$

And, on the other hand, $\nabla_{X_i}X_i$ is equal to the sum of the horizontal lift of $\nabla_{Y_i}Y_i$ and V_i, $1 \le i \le n$, where each V_i is the vertical component of $\nabla_{X_i}X_i$. Thus

$$\begin{aligned} \triangle_M(\phi\circ\pi) &= (\triangle_N\phi)\circ\pi \\ &\quad - \sum_{i=n+1}^{m} \{\text{the } \pi\text{-related horizontal component of } \nabla_{X_i}X_i\}(\phi\circ\pi). \end{aligned} \tag{1.6}$$

The Hessian of a function ϕ is the symmetric $(0,2)$ tensor field defined by

$$\operatorname{Hess}(\phi)(X,Y) = XY\phi - (\nabla_X Y)\phi, \tag{1.7}$$

and the so-called shape tensor (or "second fundamental form" tensor) of each fibre $\pi^{-1}(\sigma_n)$ is the bilinear symmetric mapping Π from $\mathcal{X}(\pi^{-1}(\sigma_n))\times\mathcal{X}(\pi^{-1}(\sigma_n))$ to $\mathcal{X}(\pi^{-1}(\sigma_n))^{\perp}$, where $\mathcal{X}(\pi^{-1}(\sigma_n))$ denotes the set of all smooth vertical vector fields of M defined on $\pi^{-1}(\sigma_n)$, such that $\Pi(X_1,X_2)$ is the component of $\nabla_{X_1}X_2$ in $\mathcal{T}(M)$ normal to the fibre $\pi^{-1}(\sigma_n)$. It turns out that

$$\begin{aligned} \operatorname{Hess}(\phi\circ\pi)(X_i,X_i) &= -\nabla_{X_i}X_i(\phi\circ\pi) \\ &= -\langle \Pi(X_i,X_i),\ \operatorname{grad}(\phi\circ\pi)\rangle, \quad n+1 \le i \le m, \end{aligned} \tag{1.8}$$

and so an equivalent expression for $\mathcal{L}$ is

$$\begin{aligned} (\mathcal{L}\phi)\circ\pi &= (\triangle_N\phi)\circ\pi + \sum_{i=n+1}^{m} \operatorname{Hess}(\phi\circ\pi)(X_i,X_i) \\ &= (\triangle_N\phi)\circ\pi - \left\langle \sum_{i=n+1}^{m} \Pi(X_i,X_i),\ \operatorname{grad}(\phi\circ\pi) \right\rangle. \end{aligned} \tag{1.9}$$

(c) Moreover, for any $(m-n)$-dimensional submanifold M_0 of M, the mean curvature vector field H_{M_0} of M_0 at $p \in M_0$ is given by

$$H_{M_0}(p) = \frac{1}{m-n} \sum_{i=n+1}^{m} \Pi(E_i,E_i), \tag{1.10}$$

where $E_{n+1},\dots,E_m$ is any orthonormal basis for the tangent space $\mathcal{T}_p(M_0)$. It is easy to check that if $x_{n+1},\dots,x_m$ is an adapted coordinate system for M_0, then

$$H_{M_0} = \frac{1}{m-n}\sum_{i,j=n+1}^{m} g_M^{ij}\Pi\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right), \tag{1.11}$$

and that if $\partial/\partial x_1,\dots,\partial/\partial x_n$ are normal to M_0, then

$$\Pi\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right) = \sum_{r=1}^{n}(\Gamma_M)_{ij}^{r}\frac{\partial}{\partial x_r}, \quad n+1 \le i,\ j \le m. \tag{1.12}$$

It follows from (1.9) that

$$(\mathcal{L}\phi)\circ\pi = (\triangle_N\phi)\circ\pi - (m-n)\langle H_{\pi^{-1}},\ \operatorname{grad}(\phi\circ\pi)\rangle, \tag{1.13}$$

which gives another expression for $\mathcal{L}$ when it exists.

So a problem rises here: what is the condition for such a differential operator $\mathcal{L}$ to exist, that is, when does the submersion π map a Brownian motion on M to a diffusion on N?

The above discussion shows that $\triangle_M(\phi\circ\pi) = (\mathcal{L}\phi)\circ\pi$ for some operator $\mathcal{L}$ on N if and only if *the traces of the second fundamental form for each fibre $\pi^{-1}(\sigma_n)$ are π-related on that fibre*; or equivalently, if and only if *the mean curvature vector fields $H_{\pi^{-1}}$ of each fibre $\pi^{-1}(\sigma_n)$ are π-related on that fibre*, for evidently either of these is the necessary and sufficient condition that $\triangle_M$ depends only on $\pi(\sigma_m)$, and not on σ_m itself.

We now discuss another condition in terms of the volume element of M for the existence of $\mathcal{L}$.

2. Some lemmas

Lemma 2.1. *Let G_M and G_N be the matrices of the local components of the metric tensor fields on M and N with respect to local coordinates $x:\sigma_m \to (x_1,\dots,x_m)$ on M and $y:\sigma_n\to(y_1,\dots,y_n)$ on N, respectively, then*

$$G_N^{-1}\circ\pi = JG_M^{-1}J^t, \tag{2.1}$$

where J is the Jacobian matrix of the coordinate representation $y\circ\pi\circ x^{-1}$ of π with the (i,j)th entry

$$\frac{\partial(y_i\circ\pi)}{\partial x_j} = \frac{\partial(y_i\circ\pi\circ x^{-1})}{\partial x_j}\circ x, \tag{2.2}$$

and J^t is its transpose.

For any given local coordinate y on N at σ_n, there exists a local coordinate x on M at $\sigma_m\in\pi^{-1}(\sigma_n)$ such that

$$y\circ\pi\circ x^{-1}:(x_1,\dots,x_m) = (y_1,\dots,y_n,z_1,\dots,z_{m-n}) \longrightarrow (y_1,\dots,y_n). \tag{2.3}$$

This implies that π is *locally* a fibration, that is, there exist a neighborhood U_n of $\sigma_n\in N$, a neighborhood U_m of $\sigma_m\in\pi^{-1}(\sigma_n)$, and a manifold F such that $\pi^{-1}(U_n)\bigcap U_m$

is diffeomorphic to $U_n \times F$ and the diffeomorphism maps $\pi^{-1}(\sigma_n)\bigcap U_m$ to F, and that the π-related vertical subspace of $\mathcal{T}_{\sigma_m}(M)$ for $\sigma_m \in \pi^{-1}(\sigma_n)$ is spanned by $\partial/\partial x_{n+1},\dots,\partial/\partial x_m$. In general, however, $\partial/\partial x_1,\dots,\partial/\partial x_n$ will not be horizontal to the π-related vertical subspace of $\mathcal{T}_{\sigma_m}(M)$.

π is called *integrable* if the horizontal distribution, which is the orthogonal complement of $\mathrm{Ker}(d\pi)$ in $\mathcal{T}(M)$, is integrable.

Lemma 2.2. *π is integrable, if and only if there exist local coordinates x and y satisfying the condition (2.3) for M and N such that the π-related horizontal subspace of $\mathcal{T}_{\sigma_m}(M)$ is spanned by $\partial/\partial x_1,\dots,\partial/\partial x_n$.*

Proof. If π-related horizontal subspace of $\mathcal{T}_{\sigma_m}(M)$ is spanned by $\partial/\partial x_1,\dots,\partial/\partial x_n$, then the horizontal distribution, by definition, is integrable.

If π is integrable, let $X_1,\dots,X_n$ be the horizontal lifts of $\partial/\partial y_1,\dots,\partial/\partial y_n$. Then the system of n differential equations in m variables

$$X_i f = 0, \quad 1 \le i \le n, \tag{2.4}$$

is complete. It follows that there are $m-n$ independent solutions $x_{n+1},\dots,x_m$ of (2.4), such that general solution of (2.4) is an arbitrary function of $x_{n+1},\dots,x_m$ (cf. [4]). Define $x_i = y_i \circ \pi$, for $1 \le i \le n$. Thus $x = (x_1,\dots,x_m)$ is the coordinate we are looking for. In fact, for any given coordinate y in N we can always find a coordinate $\tilde{x}$ in M such that (2.3) holds. Each X_i can then be formulated as

$$X_i = \frac{\partial}{\partial \tilde{x}_i} + \sum_{j=1}^{m-n} \alpha_{ij} \frac{\partial}{\partial \tilde{x}_{j+n}}, \tag{2.5}$$

where

$$(\alpha_{ij}) = E^{-1}F, \tag{2.6}$$

if the metric form of M with respect to $\tilde{x}$ is

$$\tilde{G}_M = \begin{pmatrix} E_{n\times n} & F \\ F^t & G \end{pmatrix}^{-1}. \tag{2.7}$$

Thus, the metric form of M with respect to x, by the fact that $X_i x_j = 0$, for $1 \le i \le n < j \le m$, will be

$$G_M = \begin{pmatrix} E & 0 \\ 0 & H \end{pmatrix}^{-1}, \tag{2.8}$$

for some positive definite symmetric matrix H. □

3. Main result

Proposition 3.1. *If π is integrable, then there is an operator $\mathcal{L}$ on N with*

$$(\mathcal{L}\phi) \circ \pi = \triangle_M(\phi \circ \pi), \tag{3.1}$$

if and only if the volume element $d\mu_M$ of M can be expressed as a product of two independent forms: one is a composed n-form on N with the submersion π defined by

$$\{e^{(1/2)\Phi} d\mu_N\} \circ \pi, \tag{3.2}$$

and the other is an $(m-n)$-form on the fibres $\pi^{-1}(\sigma_n)$, the local expression of which is denoted by

$$\Psi^* dx_{n+1}\cdots dx_m, \tag{3.3}$$

with the property that the latter will be independent of σ_n in a neighborhood of σ_n. And when this condition is satisfied,

$$\mathcal{L} = \triangle_N + \frac{1}{2}\operatorname{grad}\Phi. \tag{3.4}$$

Proof. The local form of the Laplace-Beltrami operator, in terms of any given coordinate x on M, is

$$\triangle_M = (\det G_M)^{-1/2}\sum_{i,j=1}^{m}\frac{\partial}{\partial x_i}\left(g_M^{ij}(\det G_M)^{1/2}\frac{\partial}{\partial x_j}\right). \tag{3.5}$$

Thus for the coordinates x and y as Lemma 2.2, we are able to obtain that, for any smooth function $\phi : N \to \mathbb{R}$,

$$\begin{aligned}
\triangle_M(\phi\circ\pi) &= \sum_{i,j=1}^{m}\left\{g_M^{ij}\frac{\partial^2}{\partial x_i\partial x_j}+\frac{\partial g_M^{ij}}{\partial x_j}\frac{\partial}{\partial x_i}+\frac{1}{2}g_M^{ij}\frac{\partial}{\partial x_j}(\log(\det G_M))\frac{\partial}{\partial x_i}\right\}(\phi\circ\pi)\\
&= \sum_{i,j=1}^{m}\left\{g_M^{ij}\sum_{k,l=1}^{n}\frac{\partial(y_k\circ\pi)}{\partial x_i}\frac{\partial(y_l\circ\pi)}{\partial x_j}\left(\frac{\partial^2\phi}{\partial y_k\partial y_l}\circ\pi\right)\right.\\
&\qquad + g_M^{ij}\sum_{k=1}^{n}\frac{\partial^2(y_k\circ\pi)}{\partial x_i\partial x_j}\left(\frac{\partial\phi}{\partial y_k}\circ\pi\right)+\frac{\partial g_M^{ij}}{\partial x_j}\sum_{k=1}^{n}\frac{\partial(y_k\circ\pi)}{\partial x_i}\left(\frac{\partial\phi}{\partial y_k}\circ\pi\right)\\
&\qquad \left.+\frac{1}{2}g_M^{ij}\frac{\partial}{\partial x_j}(\log(\det G_M))\sum_{k=1}^{n}\frac{\partial(y_k\circ\pi)}{\partial x_i}\left(\frac{\partial\phi}{\partial y_k}\circ\pi\right)\right\}\\
&= \sum_{k,l=1}^{n}g_N^{kl}\left\{\frac{\partial^2\phi}{\partial y_k\partial y_l}+\frac{1}{2}\frac{\partial}{\partial y_k}(\log(\det G_N))\frac{\partial\phi}{\partial y_l}\right\}\circ\pi\\
&\quad+\sum_{i,j=1}^{m}\left\{\frac{1}{2}g_M^{ij}\frac{\partial}{\partial x_j}\left(\log\left(\frac{\det G_M}{\det G_N\circ\pi}\right)\right)\sum_{k=1}^{n}\frac{\partial(y_k\circ\pi)}{\partial x_i}\left(\frac{\partial\phi}{\partial y_k}\circ\pi\right)\right.\\
&\qquad\left.+\sum_{k=1}^{n}\frac{\partial}{\partial x_j}\left(g_M^{ij}\frac{\partial(y_k\circ\pi)}{\partial x_i}\right)\left(\frac{\partial\phi}{\partial y_k}\circ\pi\right)\right\}\\
&= (\triangle_N\phi)\circ\pi-\left\{\sum_{k,l=1}^{n}\frac{\partial g_N^{kl}}{\partial y_l}\frac{\partial\phi}{\partial y_k}\right\}\circ\pi\\
&\quad+\sum_{i,j=1}^{m}\left\{\frac{1}{2}g_M^{ij}\frac{\partial}{\partial x_j}\left(\log\left(\frac{\det G_M}{\det G_N\circ\pi}\right)\right)\sum_{k=1}^{n}\frac{\partial(y_k\circ\pi)}{\partial x_i}\left(\frac{\partial\phi}{\partial y_k}\circ\pi\right)\right.\\
&\qquad\left.+\sum_{k=1}^{n}\frac{\partial}{\partial x_j}\left(g_M^{ij}\frac{\partial(y_k\circ\pi)}{\partial x_i}\right)\left(\frac{\partial\phi}{\partial y_k}\circ\pi\right)\right\}\\
&= (\triangle_N\phi)\circ\pi+\frac{1}{2}\sum_{j,k=1}^{n}g_N^{kj}\circ\pi\frac{\partial}{\partial x_j}\left(\log\left(\frac{\det G_M}{\det G_N\circ\pi}\right)\right)\left(\frac{\partial\phi}{\partial y_k}\circ\pi\right).
\end{aligned} \tag{3.6}$$

Note that here $\partial/\partial x_1,\dots,\partial/\partial x_n$ are the horizontal lifts of $\partial/\partial y_1,\dots,\partial/\partial y_n$. We know from the assumption that

$$\sigma_m \longrightarrow \pi\text{-related horizontal subspace of } \mathcal{T}_{\sigma_m}(M) \tag{3.7}$$

is a distribution, and

$$\sum_{j=1}^{n} g_N^{ij}\circ\pi\frac{\partial}{\partial x_j},\quad 1\le i\le n, \tag{3.8}$$

forms a basis for it. Following the same discussion as in the proof of Lemma 2.2, we know that any solution of the system of differential equations

$$\sum_{j=1}^{n} g_N^{ij}\frac{\partial f}{\partial x_j}=0,\quad 1\le i\le n, \tag{3.9}$$

is a function of $x_{n+1},\dots,x_m$. On the other hand, we have by (1.2) that existence of $\mathcal{L}$ on N if and only if there is a function Φ on N such that

$$\sum_{j=1}^{n} g_N^{ij}\circ\pi\frac{\partial}{\partial x_j}\left(\log\left(\frac{\det G_M}{\det G_N\circ\pi}\right)\right)=\left\{\sum_{j=1}^{n} g_N^{ij}\frac{\partial\Phi}{\partial y_j}\right\}\circ\pi,\quad 1\le i\le n. \tag{3.10}$$

Therefore, the existence of $\mathcal{L}$ is equivalent to that there is a function Ψ of $x_{n+1},\dots,x_m$ such that

$$\det G_M=e^{\Phi\circ\pi+\Psi(x_{n+1},\dots,x_m)}\det G_N\circ\pi. \tag{3.11}$$

$e^{\Phi}\det G_N$ is clearly a function on N. If we define a function Ψ^* on a neighborhood of $\sigma_m\in\pi^{-1}(\sigma_n)$, as the restriction of the function $e^{(1/2)\Psi}$ on $\pi^{-1}(\sigma_n)$, then Ψ^* is independent on fibres in a neighborhood of $\sigma_m\in\pi^{-1}(\sigma_n)$, and $\det G_M$ is a product of a composed function on N with π and a function on the fibres of π.

The above discussion shows that the volume element $d\mu_M$ on M is here expressed as

$$\begin{aligned} d\mu_M(\sigma_m) &= \sqrt{\det G_M}(\sigma_m)\,dx_1\cdots dx_m(\sigma_m)\\ &= \left(e^{(1/2)\Phi}\sqrt{\det G_N}\right)\circ\pi(\sigma_m)\,dy_1\cdots dy_n(\pi(\sigma_m))\\ &\quad\times\Psi^*(\sigma_m)\,dx_{n+1}\cdots dx_m(\sigma_m)\\ &= \left(e^{(1/2)\Phi}\sqrt{\det G_N}\right)(\sigma_n)\,dy_1\cdots dy_n(\sigma_n)\\ &\quad\times\Psi^*(\sigma_m)\,dx_{n+1}\cdots dx_m(\sigma_m). \end{aligned} \tag{3.12}$$

Because π is a submersion, M is locally diffeomorphic to $N\times F$ for a $(m-n)$-dimensional manifold F, and so the above condition is equivalent to that *the volume element $d\mu_M$ can locally be expressed as a product of a composed n-form on N with the submersion π and an $(m-n)$-form on F.* □

4. Remarks. (a) We know from the proof of Proposition 3.1 that, for any general coordinates such that (2.3) holds,

$$\triangle_M(\phi\circ\pi) = (\triangle_N\phi)\circ\pi+\sum_{k=1}^{n}\left\{\frac{1}{2}\sum_{j=1}^{m}g_M^{kj}\frac{\partial}{\partial x_j}\left(\log\left(\frac{\det G_M}{\det G_N\circ\pi}\right)\right)+\sum_{j=n+1}^{m}\frac{\partial g_M^{kj}}{\partial x_j}\right\}\left(\frac{\partial\phi}{\partial y_k}\circ\pi\right). \tag{4.1}$$

Compared with (1.6), we know that the kth ($1\le k\le n$) component of the vector

$$\sum_{i=n+1}^{m}\{\text{the }\pi\text{-related horizontal component of }\nabla_{X_i}X_i\} \tag{4.2}$$

is

$$-\frac{1}{2}\sum_{j=1}^{m}g_M^{kj}\frac{\partial}{\partial x_j}\left(\log\left(\frac{\det G_M}{\det G_N\circ\pi}\right)\right)-\sum_{j=n+1}^{m}\frac{\partial g_M^{kj}}{\partial x_j}. \tag{4.3}$$

And compared with (1.2), we find that there is a differential operator $\mathcal{L}$ on N with $(\mathcal{L}\phi)\circ\pi=\triangle_M(\phi\circ\pi)$ if and only if, for any $1\le k\le n$, (4.3) is a function of $\pi(\sigma_m)$, and

$$\frac{1}{2}\sum_{j=1}^{m}g_M^{kj}\frac{\partial}{\partial x_j}\left(\log\left(\frac{\det G_M}{\det G_N\circ\pi}\right)\right)+\sum_{j=n+1}^{m}\frac{\partial g_M^{kj}}{\partial x_j}=\left\{\sum_{j=1}^{n}g_N^{kj}\frac{\partial\log v}{\partial y_j}\right\}\circ\pi. \tag{4.4}$$

Therefore, for $1\le k\le n$,

$$\{\operatorname{grad}_N(\log v)\}_k\circ\pi=\frac{1}{2}\left\{\operatorname{grad}_M\log\frac{\det G_M}{\det G_N}\right\}_k+W_k, \tag{4.5}$$

where

$$W_k=\sum_{j=n+1}^{m}\frac{\partial g_m^{kj}}{\partial x_j}, \tag{4.6}$$

that is, the first n components of $\operatorname{grad}_M\{(1/2)\log(\text{volume element of the fibre }\pi^{-1}(\sigma_n))\}$ do not form a proper gradient of a function on N, which usually depend not only on $\pi(\sigma_m)$ but also on σ_m.

When

$$W_k\equiv 0,\quad 1\le k\le n. \tag{4.7}$$

Equation (4.4) can be rewritten as

$$\sum_{j=1}^{m}g_M^{kj}\frac{\partial}{\partial x_j}\left\{\log\left(\frac{\det G_M}{(v^2\det G_N)\circ\pi}\right)\right\}=0,\quad 1\le k\le n, \tag{4.8}$$

that is equivalent to

$$\left\langle dx_k,\sum_{j=1}^{m}\frac{\partial}{\partial x_j}\left\{\log\left(\frac{\det G_M}{(v^2\det G_N)\circ\pi}\right)\right\}dx_j\right\rangle=0,\quad 1\le k\le n, \tag{4.9}$$

that is,

$$\left\langle dx_k,d\left\{\log\left(\frac{\det G_M}{(v^2\det G_N)\circ\pi}\right)\right\}\right\rangle=0,\quad 1\le k\le n, \tag{4.10}$$

so that

$$d\left\{\log\left(\frac{\det G_M}{(v^2\det G_N)\circ\pi}\right)\right\} \tag{4.11}$$

is orthogonal with all dx_k for $1\le k\le n$ in $\mathcal{T}^*(M)$.

(b) When the condition in Proposition 3.1 holds, the volume element of the fibre $\pi^{-1}(\sigma_n)$ is clearly

$$e^{(1/2)\Phi(\sigma_n)}\Psi^*(x_{n+1},\dots,x_m)\,dx_{n+1}\cdots dx_m; \tag{4.12}$$

and so if π is also a fibration with compact fibre F, the $(m-n)$-dimensional volume $v(\sigma_n)$ of the fibre $\pi^{-1}(\sigma_n)$ will then be equal to

$$v(\sigma_n)=e^{(1/2)\Phi(\sigma_n)}\int_F\Psi^*(x_{n+1},\dots,x_m)dx_{n+1}\cdots dx_m=\kappa e^{(1/2)\Phi(\sigma_n)}, \tag{4.13}$$

for some constant κ, which coincides with (1.2).

(c) The condition of integrability of π in Proposition 3.1 should be able to be weakened. We study the following two cases.

(i) For the submersion π with minimal fibres, in particular with totally geodesic fibres, it is known that $\mathcal{L}=\triangle_N$, which follows immediately from the fact that the term

$$\sum_{i=n+1}^{m}\{\text{the }\pi\text{ -related horizontal component of }\nabla_{X_i}X_i\} \tag{4.14}$$

in (1.6) vanishes by the definition of minimal submanifold.

On the other hand, when M is complete and π with totally geodesic fibres, we can also obtain from the fact that (M,N,π) is a fibre bundle with the Lie group of isometries of the fibre as structure group (cf. [5] and below) that

$$d\mu_M=d\mu_N\circ\pi\times\Psi^*dx_{n+1}\cdots dx_m, \tag{4.15}$$

for a suitable coordinate $(x_{n+1},\dots,x_m)$ on fibres.

In the case that π is with minimal fibres, it follows from the fact that the structure group of the bundle (which is a priori the group of diffeomorphisms of the fibre F) reduces to the group of volume preserving diffeomorphisms of F (cf. [1]) that the volume element of M is of the expression (4.15).

(ii) The case that the submersion π is a quotient mapping with respect to a Lie group G of isometries acting properly and freely on M.

The fibre $\pi^{-1}(\sigma_n)$ here inherits a Riemannian structure from that of M, and the corresponding volume element $d\mu_{\pi^{-1}(\sigma_n)}$ of the fibre $\pi^{-1}(\sigma_n)$ is invariant under G by the transitive action of G of isometries on the fibres. Under the identification $\pi^{-1}(\sigma_n)=G$, the volume elements $d\mu_{\pi^{-1}(\sigma_n)}$ and dg, the unique left-invariant volume element up to constants of G, must, by the uniqueness, be proportional (cf. [2]). Hence there exists a function $e^{(1/2)\Phi}$ on N such that

$$d\mu_{\pi^{-1}(\sigma_n)}=e^{(1/2)\Phi(\sigma_n)}dg, \tag{4.16}$$

and so

$$d\mu_M=dg\{e^{(1/2)\Phi}d\mu_N\}\circ\pi, \tag{4.17}$$

which gives a form for the volume element on M coincident with our claim if we notice that here M is locally diffeomorphic to $N\times G$.

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