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# SOME INEQUALITIES IN $B(H)$

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ABSTRACT. Let $H$ denote a separable Hilbert space and let $B(H)$ be the space of bounded and linear operators from $H$ to $H$. We define a subspace $\Delta(A,B)$ of $B(H)$, and prove two inequalities between the distance to $\Delta(A,B)$ of each operator $T$ in $B(H)$, and the value $\sup\{\|A^nTB^n - T\| : n = 1,2,\ldots\}$.

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**1. Notations.** Throughout this paper $H$ denotes a separable Hilbert space and $\{e_n\}_{n=1}^{\infty}$ an orthonormal basis. Let $L_A$ and $R_B$ be left and right translation operators on $B(H)$ for $A,B \in B(H)$, satisfying $\|A\| \le 1$ and $\|B\| \le 1$. Then the set $\Delta(A,B)$ is defined by

$$\Delta(A,B) = \{T \in B(H) : ATB = T\} = \{T \in B(H) : ST = T\}, \tag{1.1}$$

where $S = L_A R_B$.

An operator $C \in B(H)$ is called positive, if $\langle Cx, x\rangle \ge 0$ for all $x \in H$. Then for any positive operator $C \in B(H)$ we define $\operatorname{tr} C = \sum_{n=1}^{\infty}\langle e_n, Ce_n\rangle$. The number $\operatorname{tr} C$ is called the trace of $C$ and is independent of the orthonormal basis chosen. An operator $C \in B(H)$ is called trace class if and only if $\operatorname{tr}|C| < \infty$ for $|C| = (C^*C)^{1/2}$, where $C^*$ is adjoint of $C$. The family of all trace class operators is denoted by $L_1(H)$. The basic properties of $L_1(H)$ and the functional $\operatorname{tr}(\cdot)$ are the following:

(i) Let $\|\cdot\|_1$ be defined in $L_1(H)$ by $\|C\|_1 = \operatorname{tr}|C|$. Then $L_1(H)$ is a Banach space with the norm $\|\cdot\|_1$ and $\|C\| \le \|C\|_1$.

(ii) $L_1(H)$ is $*$- ideal, that is,
  (a) $L_1(H)$ is a linear space,
  (b) if $C \in L_1(H)$ and $D \in B(H)$, then $CD \in L_1(H)$ and $DC \in L_1(H)$,
  (c) if $C \in L_1(H)$, then $C^* \in L_1(H)$.

(iii) $\operatorname{tr}(\cdot)$ is linear.

(iv) $\operatorname{tr}(CD) = \operatorname{tr}(DC)$ if $C \in L_1(H)$ and $D \in B(H)$.

(v) $B(H) = L_1(H)^*$, that is, the map $T \to \operatorname{tr}(T)$ is an isometric isomorphism of $B(H)$ onto $L_1(H)^*$, (see [3]).

Let $X$ be a Banach space. If $M \subset X$, then

$$M^{\perp} = \{x^* \in X^* : \langle x, x^*\rangle = 0,\ x \in M\} \tag{1.2}$$

is called the annihilator of $M$. If $N \subset X^*$, then

$$^{\perp}N = \{x \in X : \langle x, x^*\rangle = 0,\ x^* \in N\} \tag{1.3}$$

is called the preannihilator of $N$. Rudin [4] proved for these subspaces:

(i) ${}^{\perp}(M^{\perp})$ is the norm closure of $M$ in $X$.

(ii) $({}^{\perp}N)^{\perp}$ is the weak-$*$ closure of $N$ in $X^*$.

## 2. Main results

**Lemma 2.1.** *Let $X$ be a Banach space. If $P$ is a continuous operator in the weak-$*$ topology on the dual space $X^*$, then there exists an operator $T$ on $X$ such that $P = T^*$.*

**Proof.** If $P : X^* \to X^*$, then $P^* : X^{**} \to X^{**}$. We know that the continuous functionals in the weak-$*$ topology on $X^*$ are simply elements of $X$, (see [4]). Then we must show that $P^*x$ is continuous in the weak-$*$ topology on $X^*$ for all $x \in X$. Let $(x'_n)$ be a sequence in $X^*$ such that $x'_n \to x'$, $x' \in X^*$. Then we have

$$\langle P^*x, x'_n\rangle = \langle x, Px'_n\rangle \longrightarrow \langle x, Px'\rangle = \langle P^*x, x'\rangle. \tag{2.1}$$

Hence $P^*x$ is continuous in the weak-$*$ topology on $X^*$ for all $x \in X$, so $P^*x \in X$. If $T$ is the restriction to $X$ of $P^*$, then we have

$$\langle x, T^*x'\rangle = \langle Tx, x'\rangle = \langle P^*x, x'\rangle = \langle x, Px'\rangle \tag{2.2}$$

for all $x \in X$ and $x' \in X^*$. Hence $P = T^*$. □

**Definition 2.2.** If $P_*$ is the operator $T$ in Lemma 2.1, then $P_*$ is called the preadjoint operator of $P$.

The operator $x \otimes y \in B(H)$ for each $x, y \in H$ is defined by $(x \otimes y)z = \langle z, y\rangle x$ for all $z \in H$. It is easy to see that this operator has the following properties:

(i) $T(x \otimes y) = Tx \otimes y$.

(ii) $(x \otimes y)T = x \otimes T^*y$.

(iii) $\operatorname{tr}(x \otimes y) = \langle y, x\rangle$.

The following lemma is an easy application of some properties of the operator $x \otimes y$ $(x, y \in H)$ and the functional $\operatorname{tr}(\cdot)$.

**Lemma 2.3.** (i) *Suppose $K$ is a closed subset in the weak-$*$ topology of $B(H)$. Then $K$ is closed in the weak-$*$ topology of $B(H)$.*

(ii) *$S = L_A R_B$ is continuous in the weak-$*$ topology of $B(H)$ for all $A, B \in B(H)$, satisfying $\|A\| \le 1$ and $\|B\| \le 1$.*

**Lemma 2.4.** *There exists a linear subspace $M$ of $L_1(H)$ such that $\Delta(H) = M^{\perp}$ and $M$ is closed linear span of $\{S_*X - X : X \in L_1(H)\}$, where $S_*$ is the preadjoint operator of $S$.*

**Proof.** Note that

$${}^{\perp}\Delta(A,B) = \{U \in L_1(H) : \langle U, U^*\rangle = 0,\ U^* \in \Delta(A,B)\}. \tag{2.3}$$

It is known that $({}^{\perp}\Delta(A,B)^{\perp})$ is the weak-$*$ closure of $\Delta(A,B)$ (see [4]). Then we can write $({}^{\perp}\Delta(A,B))^{\perp} = \Delta(A,B)$, since $\Delta(A,B)$ is a closed set in the weak-$*$ topology of $B(H)$. We say ${}^{\perp}\Delta(A,B) = M$. Now we show that $M$ is the closed linear span of $\{S_*U - U : U \in L_1(H)\}$. For this, it is sufficient to prove that $\langle S_*U - U, T\rangle = 0$ for all $T \in \Delta(A,B)$.

Indeed since $ST = T$, we have

$$\langle S_* X - X, T\rangle = \langle (S_* - I)X, T\rangle = \langle X, (S_* - I)^* T\rangle = \langle X, (S - I)T\rangle = 0. \tag{2.4}$$

□

**Lemma 2.5.** *Let $K(T)$ be the closed convex hull of $\{S^n T : n = 1,2,\dots\}$ in the weak operator topology, for a fixed $T \in B(H)$. Then we have*

$$K(T) \cap \Delta(A,B) \neq 0. \tag{2.5}$$

**Proof.** Assume $K(T) \cap \Delta(A,B) = 0$. By Lemma 2.3, $K(T)$ is closed in the weak-$*$ topology. It is easy to see that $K(T)$ is bounded. Then $K(T)$ is compact in the weak-$*$ topology by Alaoglu, [1]. Since $S$ is continuous in the weak-$*$ topology, if $U_\alpha \to U$ for $(U_\alpha)_{\alpha\in I} \subset \Delta(A,B)$, then $SU_\alpha = U_\alpha \to SU$. Hence $\Delta(A,B)$ is closed in the weak-$*$ topology. This shows that $U \in \Delta(A,B)$.

Since $K(T)$ is compact and convex in the weak-$*$ topology, and $\Delta(A,B)$ is closed in the weak-$*$ topology, and $K(T) \cap \Delta(A,B) = 0$, there exist some $U_0 \in M$ and $\sigma > 0$ such that

$$|\operatorname{tr}(TU_0)| \geq \sigma \tag{2.6}$$

for all $T \in \Delta(A,B)$, (see [2]). Now we define the operators $T_n \sum_{k=1}^{n} S^k T$ for all positive integer $n$. These operators are clearly in $K(T)$. It is easy to show that the operators $T_n$ is bounded. Also by Lemma 2.4, there is a $U \in L_1(H)$ such that $U_0 = S_* U - U$. Then we have

$$\begin{aligned} |\langle T_n, U_0\rangle| &= |\langle T_n, S_* U - U\rangle| = |\langle ST_n, U\rangle - \langle T_n, U\rangle| \\ &= \left|\left\langle S\left(\frac{1}{n}\sum_{k=1}^{n} A^k T B^k\right), U\right\rangle - \left\langle \frac{1}{n}\sum_{k=1}^{n} A^k T B^k, U\right\rangle\right| \\ &= \left|\left\langle \frac{1}{n}\sum_{k=1}^{n} A^{k+1} T B^{k+1}, U\right\rangle - \left\langle \frac{1}{n}\sum_{k=1}^{n} A^k T B^k, U\right\rangle\right| \\ &= \frac{1}{n}|\langle A^{n+1} T B^{n+1} - ATB, U\rangle| \\ &\leq \frac{1}{n} 2\|T\| \cdot \|U\|. \end{aligned} \tag{2.7}$$

This implies that $|\langle T_n, X_0\rangle| \to 0$, which is a glaring contradiction to (2.6). □

**Theorem 2.6.** *Let $H$ be separable Hilbert space and $T \in B(H)$. Then we have*

(i) $d(T, \Delta(A,B)) \geq (1/2)\sup_n \|S^n T - T\|$,

(ii) $d(T, \Delta(A,B)) \leq \sup_n \|S^n T - T\|$.

**Proof.** (i) We can write

$$S^n T - T = S^n(T - T_0) - (T - T_0) + S^n T_0 - T_0 \tag{2.8}$$

for each $T_0 \in \Delta(A,B)$. Hence we have

$$||S^n T - T|| \leq ||S^n||||T - T_0|| + ||T - T_0|| \leq 2||T - T_0||. \tag{2.9}$$

This shows that

$$\frac{1}{2}\sup_n ||S^nT - T|| \leq \inf_{T_0\in\Delta(A,B)} ||T - T_0||. \tag{2.10}$$

The inequality (2.10) gives that

$$d(T,\Delta(A,B)) \geq \frac{1}{2}\sup_n ||S^nT - T||. \tag{2.11}$$

(ii) Let $K(T)$ be as Lemma 2.5. Then we can write

$$K(T) = \text{co}\{S^nT : n = 1,2,\dots\}. \tag{2.12}$$

Now take any element $U = \sum_{k=1}^n \lambda_k S^k T$ in the set $\text{co}\{S^nT : n = 1,2,\dots\}$, where $\sum_{k=1}^n \lambda_k = 1$, $\lambda_k \geq 0$. Then

$$\begin{aligned} \|U - T\| &= \left\|\sum_{k=1}^n \lambda_k S^k T - T\right\| \leq \left\|\sum_{k=1}^n \lambda_k S^k T - \sum_{k=1}^n \lambda_k T\right\| \\ &\leq \sum_{k=1}^n \lambda_k ||S^kT - T|| \leq \sum_{k=1}^n \lambda_k \sigma(T) = \sigma(T), \end{aligned} \tag{2.13}$$

where $\sigma(T) = \sup_n \|S^nT - T\|$. That is, for all $U \in \text{co}\{S^nT : n = 1,2,\dots\}$ is

$$\|U - T\| \leq \sup_n ||S^nT - T||. \tag{2.14}$$

Since there is a sequence $(U_n)$ in $\text{co}\{S^nT : n = 1,2,\dots\}$ such that $U_n \to V$ for all $V \in K(T)$, then we write

$$\|V - T\| \leq ||V - T_n|| + ||T_n - T||. \tag{2.15}$$

If we use the inequalities (2.14) and (2.15), we easily see that

$$\|V - T\| \leq \sup_n ||S^nT - T||. \tag{2.16}$$

Also since $K(T) \cap \Delta(A,B) \neq 0$ by Lemma 2.5, then we obtain

$$||T - T_0|| \leq \sup_n ||S^nT - T|| \tag{2.17}$$

for a $T_0 \in K(T) \cap \Delta(A,B)$. Hence we can write

$$d(T,\Delta(A,B)) = \inf_{U\in\Delta(A,B)} \|T - U\| \leq ||T - T_0|| \leq \sup_n ||S^nT - T||. \tag{2.18}$$

This completes the proof. □

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