IJMMS 25:7 (2001) 467–478
PII. S0161171201002381
http://ijmms.hindawi.com

ON HENSTOCK-DUNFORD AND HENSTOCK-PETTIS INTEGRALS

YE GUOJU and AN TIANQING

(Received 22 October 1998)

Abstract. We give the Riemann-type extensions of Dunford integral and Pettis integral, Henstock-Dunford integral and Henstock-Pettis integral. We discuss the relationships between the Henstock-Dunford integral and Dunford integral, Henstock-Pettis integral and Pettis integral. We prove the Harnack extension theorems and the convergence theorems for Henstock-Dunford and Henstock-Pettis integrals.

2000 Mathematics Subject Classification. Primary 26A39, 28B05; Secondary 28B20, 46G10, 46G12.

1. Introduction. During 1957–1958, R. Henstock and J. Kurzweil, independently, gave a Riemann-type integral called the Henstock-Kurzweil integral (or Henstock integral) (see [7]). It is a kind of nonabsolute integral and contains the Lebesgue integral. It has been proved that this integral is equivalent to the special Denjoy integral [7]. The Dunford, Pettis integrals are generalizations of the Lebesgue integral to Banach-valued functions. In [5], R. A. Gordon gave two Denjoy-type extensions of the Dunford, Pettis integrals, the Denjoy-Dunford and Denjoy-Pettis integrals, and discussed their properties.

In this paper, we give the Riemann-type extensions of Dunford, Pettis integrals, the Henstock-Dunford, Henstock-Pettis integrals, and discuss the relationships between the Henstock-Dunford integral and Dunford integral, Henstock-Pettis integral and Pettis integral. We prove the Harnack extension theorems and the convergence theorems for Henstock-Dunford and Henstock-Pettis integrals.

Throughout this paper, X denotes a real Banach space and X^* its dual. $B(X^*) = \{x^* \in X^* : \| x^* \| \le 1\}$ is the unit ball in X^*. $I_0 = [a,b]$ is a closed interval in $\mathbb{R}$.

We first give some preliminaries. A partition D of $[a,b]$ is a finite collection of interval-point pairs (I,t) with the intervals nonoverlapping and their union $[a,b]$. Here t is the associated point of I. We write $D = \{(I,t)\}$, it is said to be δ-fine partition of $[a,b]$ if for each interval-point pair (I,t), we have $t \in I \subset (t-\delta(t), t+\delta(t))$.

Definition 1.1 (see [7]). A function $f : [a,b] \to \mathbb{R}$ is Henstock integrable if there exists a function $F : [a,b] \to \mathbb{R}$ such that for every $\epsilon > 0$ there is a function $\delta(t) > 0$ such that for any δ-fine partition $D = \{[u,v];t\}$ of $[a,b]$, we have

$$\left| \sum \left[f(t)(v-u) - F(u,v) \right] \right| < \epsilon, \tag{1.1}$$

where the sum $\sum$ is understood to be over $D = \{([u,v],t)\}$ and $F(u,v) = F(v) - F(u)$. We write $(H)\int_{I_0} f = F(I_0)$.

The function f is said to be Henstock integrable on the set $E \subset [a,b]$ if the function $f\chi_E$ is Henstock integrable on $[a,b]$. We write $(H)\int_{I_0} f\chi_E = (H)\int_E f$.

Definition 1.2 (see [1, 5, 7]). A function $f : [a,b] \to \mathbb{R}$ is Denjoy (or special Denjoy) integrable if there exists an ACG (or ACG^*) function $F : [a,b] \to \mathbb{R}$ such that $D_{\text{ap}}F(t) = f(t)$ (or $F'(t) = f(t)$) almost everywhere on $[a,b]$. Where $D_{\text{ap}}F(t)$ denotes the approximate derivative of F at t. We write $(D)\int_{I_0} f = F(I_0)$ (or $(D^*)\int_{I_0} f = F(I_0)$).

The function f is said to be Denjoy (or special Denjoy) integrable on the set $E \subset [a,b]$ if the function $f\chi_E$ is Denjoy (or special Denjoy) integrable on $[a,b]$. We write $(D)\int_{I_0} f\chi_E = (D)\int_E f$ (or $(D^*)\int_{I_0} f\chi_E = (D^*)\int_E f$).

If f is special Denjoy integrable, then f is Denjoy integrable.

Lemma 1.3 (see [7]). *A function $f : [a,b] \to \mathbb{R}$ is Henstock integrable on $[a,b]$ if and only if f is the special Denjoy integrable on $[a,b]$.*

Definition 1.4 (see Gordon [5]). (a) A function $f : [a,b] \to X$ is Denjoy-Dunford integrable on $[a,b]$ if for each x^* in X^* the function x^*f is Denjoy integrable on $[a,b]$ and if for every interval I in $[a,b]$ there exists a vector x_I^{**} in X^{**} such that $x_I^{**}(x^*) = \int_I x^*f$ for all x^* in X^*. We write $x_{I_0}^{**} = (DD)\int_{I_0} f = F(I_0)$ and F is called the primitive of f on I_0.

(b) A function $f : [a,b] \to X$ is Denjoy-Pettis integrable on $[a,b]$ if f is Denjoy-Dunford integrable on $[a,b]$ and if $x_I^{**} \in X$ for every interval I in $[a,b]$. We write $x_{I_0}^{**} = (DP)\int_{I_0} f = F(I_0)$ and F is called the primitive of f on I_0.

The function f is said to be integrable in one of the above senses on the set $E \subset [a,b]$ if the function $f\chi_E$ is integrable in that sense on $[a,b]$.

Lemma 1.5 (see [3]). *A function $f : [a,b] \to X$ is Denjoy-Dunford integrable on $[a,b]$ if and only if x^*f is Denjoy integrable on $[a,b]$ for all $x^* \in X^*$.*

2. Definition and properties.

In the following, we give the Riemann-type extensions of Dunford, Pettis integrals, and discuss the relationships between Henstock-Dunford integral and Dunford integral, Henstock-Pettis integral and Pettis integral.

Definition 2.1. (a) A function $f : [a,b] \to X$ is Henstock-Dunford integrable on $[a,b]$ if for each x^* in X^* the function x^*f is Henstock integrable on $[a,b]$ and if for every interval I in $[a,b]$ there exists a vector x_I^{**} in X^{**} such that $x_I^{**}(x^*) = \int_I x^*f$ for all x^* in X^*. We write $x_{I_0}^{**} = (HD)\int_{I_0} f = F(I_0)$ and F is called the primitive of f on I_0.

(b) A function $f : [a,b] \to X$ is Henstock-Pettis integrable on $[a,b]$ if f is Henstock-Dunford integrable on $[a,b]$ and if $x_I^{**} \in X$ for every interval I in $[a,b]$. We write $x_{I_0}^{**} = (HP)\int_{I_0} f = F(I_0)$ and F is called the primitive of f on I_0.

The function f is said to be integrable in one of the above senses on the set $E \subset [a,b]$ if the function $f\chi_E$ is integrable in that sense on $[a,b]$.

By the above definitions and Definition 1.4, it is easy to see that if f is Henstock-Dunford (or Henstock-Pettis) integrable on I_0, then f is Denjoy-Dunford (or Denjoy-Pettis) integrable.

Theorem 2.2. *A function $f:[a,b]\to X$ is Henstock-Dunford integrable on $[a,b]$ if and only if x^*f is Henstock integrable on $[a,b]$ for all $x^*\in X^*$.*

Proof. If f is Henstock-Dunford integrable on $[a,b]$, for every $x^*\in X^*$, by Definition 2.1, x^*f is Henstock integrable on $[a,b]$. Conversely, if x^*f is Henstock integrable on $[a,b]$. It follows from Lemma 1.3 that x^*f is Denjoy integrable on $[a,b]$ and $(D)\int_a^b x^*f=(H)\int_a^b x^*f$. It follows from Lemma 1.5 that f is Denjoy-Dunford integrable on $[a,b]$, and for every interval I in $[a,b]$ there exists a vector x_I^{**} in X^{**} such that $x_I^{**}(x^*)=(D)\int_I x^*f$ for all x^* in X^*, that is, $x_I^{**}(x^*)=(H)\int_I x^*f$ for all x^* in X^*. So f is Henstock-Dunford integrable on $[a,b]$. □

Theorem 2.3. *If the function $f:[a,b]\to X$ is Henstock-Dunford integrable on $[a,b]$, then each perfect set in $[a,b]$ contains a portion on which f is Dunford integrable.*

Proof. Since the function $f:[a,b]\to X$ is Henstock-Dunford integrable on $[a,b]$, then for each $x^*\in X^*$, x^*f is Henstock integrable on $[a,b]$. It follows from [8] that each perfect set in $[a,b]$ contains a portion on which x^*f is Lebesgue integrable. So f is Dunford integrable on a portion. □

Theorem 2.4. *If the function $f:[a,b]\to X$ is Henstock-Dunford integrable on $[a,b]$, then there is a sequence $\{X_k\}$ of closed subsets such that $X_k\subset X_{k+1}$ for all k, $\cup_{k=1}^{\infty}X_K=[a,b]$, f is Dunford integrable on each X_k and*

$$\lim_{k\to\infty}(\textit{Dunford})\int_{X_k\cap[a,x]}f(t)\,dt=(HD)\int_a^x f(t)\,dt\ \textit{weakly}\tag{2.1}$$

uniformly on $[a,b]$.

Proof. It follows from Theorem 2.2 that a function $f:[a,b]\to X$ is Henstock-Dunford integrable on $[a,b]$ if and only if x^*f is Henstock integrable on $[a,b]$ for all $x^*\in X^*$. From [8], x^*f is Henstock integrable on $[a,b]$, then there is a sequence $\{X_k\}$ of closed subsets such that $X_k\subset X_{k+1}$ for all k, $\cup_{k=1}^{\infty}X_K=[a,b]$, x^*f is Lebesgue integrable on each X_k and

$$\lim_{k\to\infty}(L)\int_{X_k\cap[a,x]}x^*f(t)\,dt=(H)\int_a^x x^*f(t)\,dt\tag{2.2}$$

uniformly on $[a,b]$ for each $x^*\in X^*$. So we obtain that f is Dunford integrable on each X_k and

$$\lim_{k\to\infty}(\text{Dunford})\int_{X_k\cap[a,x]}f(t)\,dt=(HD)\int_a^x f(t)\,dt\ \text{weakly}\tag{2.3}$$

uniformly on $[a,b]$. □

Theorem 2.5. *If the function $f:[a,b]\to X$ is Henstock-Dunford integrable on $[a,b]$, then there exists a sequence $\{X_k\}$ of closed sets, $\cup_{k=1}^{\infty}X_k=[a,b]$, f is Dunford integrable on each X_k.*

Proof. Since f Henstock-Dunford integrable on $[a,b]$, by Definition 2.1, for each $x^*\in X^*$, x^*f is Henstock integrable on $[a,b]$, and for every interval $I\subset[a,b]$,

$\int_I x^* f = x^* \int_I f$, and $F(I) = \int_I f \in X$. Since $x^* f$ is Henstock integrable, then $x^* F$ is ACG^*. So there is a sequence $\{X_k\}$ of closed subsets such that $\cup_{k=1}^{\infty} X_k = [a,b]$ and $x^* F$ is VB^* on each X_k. From [7, Lemma 6.18], $x^* f$ is Lebesgue integrable on each X_k. So we obtain that f is Dunford integrable on each X_k. □

Theorem 2.6. *Suppose that X contains no copy of c_0 and $f : [a,b] \to X$. If the function f is Henstock-Pettis integrable on $[a,b]$, then each perfect set in $[a,b]$ contains a portion on which f is Pettis integrable.*

Proof. Since the function $f : [a,b] \to X$ is Henstock-Pettis integrable on $[a,b]$, then f is Denjoy-Pettis integrable on $[a,b]$. It follows from [5, Theorem 38] that each perfect set in $[a,b]$ contains a portion on which f is Pettis integrable.

In the fact, from [3, Theorem 10], we have that if each Henstock-Pettis integrable function defined on $[a,b]$ is Pettis integrable on a portion of every close set, then X does not contain c_0. □

Theorem 2.7. *Suppose that X contains no copy of c_0 and $f : [a,b] \to X$ is a measurable. If the function $f : [a,b] \to X$ is Henstock-Pettis integrable on $[a,b]$, then there exists a sequence $\{X_k\}$ of closed sets with $X_k \uparrow [a,b]$ such that for each $x^* \in X^*$, f is Pettis integrable on each X_k, and*

$$\lim_{k\to\infty} (\mathit{Pettis}) \int_{X_k} f = (HP) \int_a^b f \ \mathit{weakly}. \tag{2.4}$$

Proof. Since f is Henstock-Pettis integrable on $[a,b]$, then f is Henstock-Dunford integrable on $[a,b]$. By Theorem 2.4, there is a sequence $\{X_k\}$ of closed subsets such that $X_k \subset X_{k+1}$ for all k, $\cup_{k=1}^{\infty} X_K = [a,b]$, $x^* f$ is Lebesgue integrable on each X_k and

$$\lim_{k\to\infty} (L) \int_{X_k \cap [a,x]} x^* f(t)\,dt = (H) \int_a^x x^* f(t)\,dt \tag{2.5}$$

uniformly on $[a,b]$ for each $x^* \in X^*$. So we obtain that f is Dunford integrable on each X_k. From [2, Theorem 2.5, page 54], f is Pettis integrable on $[a,b]$ and

$$\lim_{k\to\infty} (\text{Pettis}) \int_{X_k \cap [a,x]} f(t)\,dt = (HP) \int_a^x f(t)\,dt \ \text{weakly} \tag{2.6}$$

uniformly on each X_k, that is,

$$\lim_{k\to\infty} (\text{Pettis}) \int_{X_k} f = (HP) \int_a^b f \ \text{weakly}. \tag{2.7}$$

□

In Theorem 2.7, if we remove the condition that f is a measurable, then we have the following theorem.

Theorem 2.8. *Suppose that X contains no copy of c_0. If the function $f : [a,b] \to X$ is Henstock-Pettis integrable on $[a,b]$, then there exists a sequence $\{X_k\}$ of closed sets, $\cup_{k=1}^{\infty} X_k = [a,b]$, f is Pettis integrable on each X_k.*

Proof. Since f is Henstock-Pettis integrable on $[a,b]$, by Definition 2.1, for each $x^* \in X^*$, $x^* f$ is Henstock integrable on $[a,b]$, and for every interval $I \subset [a,b]$,

$\int_I x^*f = x^*\int_I f$, and $F(I) = \int_I f \in X$. Since x^*f is Henstock integrable, then x^*F is ACG^*. So there is a sequence $\{X_k\}$ of closed subsets such that $\cup_{k=1}^{\infty} X_k = [a,b]$ and x^*F is VB^* on each X_k. For each $k \in N$, let $(a,b) - X_k = \cup_{n=1}^{\infty}(c_n^k, d_n^k)$. Then

$$\sum_{n=1}^{\infty}\left|x^*\int_{c_n^k}^{d_n^k} f\right| < \infty. \tag{2.8}$$

Since X contains no copy of c_0, by Bessaga-Pelczynski theorem [2, page 22], $\sum_{n=1}^{\infty}\int_{c_n^k}^{d_n^k} f$ is unconditionally convergent in norm. Also

$$\sum_{n=1}^{\infty}\sup_{[a_n^k,b_n^k]\subset[c_n^k,d_n^k]}\left|x^*\int_{a_n^k}^{b_n^k} f\right| < \infty. \tag{2.9}$$

By Harnack extension theorem [7, page 41], we have

$$\int_{X_k} x^*f = \int_a^b x^*f - \sum_{n=1}^{\infty}\int_{c_n^k}^{d_n^k} x^*f = x^*\left(\int_a^b f - \sum_{n=1}^{\infty}\int_{c_n^k}^{d_n^k} f\right). \tag{2.10}$$

Hence $\int_{X_k} f = \int_a^b f - \sum_{n=1}^{\infty}\int_{c_n^k}^{d_n^k} f \in X$ and $\int_{X_k} x^*f = x^*\int_{X_k} f$.

So, for every closed set $H \subset X_k$, we have $\int_H x^*f = x^*\int_H f$ and $\int_H f \in X$. Since $\int_a^b f\chi_{X_k} = \int_{X_k} f \in X$, $\int_a^b f\chi_H = \int_H f \in X$, then for every closed interval $I \subset [a,b]$, $\int_I f\chi_{X_k} = \int_{I\cap X_k} f \in X$. By [5, Theorem 23, page 79], $f\chi_{X_k}$ is Pettis integrable on $[a,b]$, that is, f is Pettis integrable on each X_k. □

3. The extension theorems and convergence theorems. Now we consider the extension theorems and convergence theorems of the Henstock-Dunford and Henstock-Pettis integrals.

Theorem 3.1. *Let E be a closed subset in $[a,b]$ and $(a,b) - E$ the union of $\{(a_k,b_k)\}$, $k = 1,2,\ldots$. If $f : [a,b] \to X$ is Henstock-Dunford integrable on E and each interval $[a_k,b_k]$ with*

$$\sum_{k=1}^{\infty}\omega\left(\int_{a_k}^{t} x^*f, [a_k,b_k]\right) < \infty \tag{3.1}$$

for each $x^* \in X^*$, then f is Henstock-Dunford integrable on $[a,b]$ and

$$\left\langle x^*, (HD)\int_a^b f\right\rangle = \left\langle x^*, (HD)\int_a^b f\chi_E\right\rangle + \sum_{k=1}^{\infty}\left\langle x^*, (HD)\int_{a_k}^{b_k} f\right\rangle \tag{3.2}$$

for each $x^* \in X^*$.

Proof. From the conditions of Theorem 3.1, we have the function x^*f satisfies the hypothesis of [7, Corollary 7.11]. So we have x^*f is Henstock integrable on $[a,b]$ and

$$(H)\int_a^b x^*f = (H)\int_a^b x^*f\chi_E + \sum_{k=1}^{\infty}(H)\int_{a_k}^{b_k} x^*f. \tag{3.3}$$

It follows from Theorem 2.2 that f is Henstock-Dunford integrable on $[a,b]$ and the above equality means that

$$\left\langle x^*,(HD)\int_a^b f\right\rangle = \left\langle x^*,(HD)\int_a^b f\chi_E\right\rangle + \sum_{k=1}^{\infty}\left\langle x^*,(HD)\int_{a_k}^{b_k} f\right\rangle \tag{3.4}$$

for each $x^* \in X^*$. □

Theorem 3.2. *Let E be a closed subset in $[a,b]$ and $\{(a_k,b_k)\}$ be an enumeration of the intervals contiguous to E in (a,b). Suppose that $f:[a,b]\to X$ is Henstock-Pettis integrable on E and each interval $[a_k,b_k]$. If $\sum_{k=1}^{\infty}\omega(\int_{a_k}^{t}x^*f,[a_k,b_k])<\infty$ for each $x^*\in X^*$ and the series $\sum_{k=1}^{\infty}(HP)\int_{[a_k,b_k]\cap J} f$ is unconditionally convergent for every subinterval J of $[a,b]$, then f is Henstock-Pettis integrable on $[a,b]$ and*

$$(HP)\int_a^b f = (HP)\int_a^b f\chi_E + \sum_{k=1}^{\infty}(HP)\int_{a_k}^{b_k} f. \tag{3.5}$$

Proof. From Theorem 3.1, we have the function f is Henstock-Dunford integrable on $[a,b]$ and $(H)\int_a^b x^*f = (H)\int_a^b x^*f\chi_E + \sum_{k=1}^{\infty}(H)\int_{a_k}^{b_k} x^*f$. To show that f is in fact Henstock-Pettis integrable on $[a,b]$. We need to show that $(HD)\int_J f$ belongs to X for each closed interval J in $[a,b]$.

Let $E_0 = E\cap J$. Then E_0 is a closed set. Since $f\chi_E$ is Henstock-Pettis integrable on J, then $f\chi_{E_0}$ is Henstock-Pettis integrable on J, that is, f is Henstock-Pettis integrable on E_0. And $\{(a_k,b_k)\cap J\}$ is an enumeration of the intervals contiguous to E_0 in J, so f is Henstock-Pettis integrable on them and $\sum_k (HP)\int_{[a_k,b_k]\cap J} f$ is an unconditionally convergent series in X. Now, if we apply Theorem 3.1 to E_0 in J, we get

$$\left\langle x^*,(HD)\int_J f\right\rangle = \left\langle x^*,(HP)\int_J f\chi_{E_0}\right\rangle + \sum_{k=1}^{\infty}\left\langle x^*,(HP)\int_{[a_k,b_k]\cap J} f\right\rangle \tag{3.6}$$

for each $x^*\in X^*$, that is,

$$\left\langle x^*,(HD)\int_J f\right\rangle = \left\langle x^*,(HP)\int_J f\chi_{E_0} + \sum_{k=1}^{\infty}(HP)\int_{[a_k,b_k]\cap J} f\right\rangle \tag{3.7}$$

for each $x^*\in X^*$. We conclude that

$$(HD)\int_J f = (HP)\int_J f\chi_{E_0} + \sum_{k=1}^{\infty}(HP)\int_{[a_k,b_k]\cap J} f. \tag{3.8}$$

Hence, f is Henstock-Pettis integrable on $[a,b]$ and

$$(HP)\int_a^b f = (HP)\int_a^b f\chi_{E_0} + \sum_{k=1}^{\infty}(HP)\int_{[a_k,b_k]\cap J} f. \tag{3.9}$$

□

Corollary 3.3. *Suppose that X contains no copy of c_0. Let E be a closed subset in $[a,b]$ and $\{(a_k,b_k)\}$ be an enumeration of the intervals contiguous to E in (a,b). Suppose that $f:[a,b]\to X$ is Henstock-Pettis integrable on E and each interval $[a_k,b_k]$.*

If $\sum_{k=1}^{\infty}\omega(\int_{a_k}^{t}x^*f,[a_k,b_k])<\infty$ *for each* $x^*\in X^*$*, then* f *is Henstock-Pettis integrable on* $[a,b]$ *and*

$$(HP)\int_a^b f=(HP)\int_a^b f\chi_E+\sum_{k=1}^{\infty}(HP)\int_{a_k}^{b_k}f. \tag{3.10}$$

Theorem 3.4. *Suppose that* X *is weakly sequentially complete and* $f:[a,b]\to X$ *is Henstock-Dunford integrable on* $[a,b]$*. If* f *is measurable, then* f *is Henstock-Pettis integrable on* $[a,b]$*.*

Proof. It is similar to the proof of [5, Theorem 40]. □

Lemma 3.5 (see [1, 5])**.** *Let* Γ *be a family of open intervals in* (a,b) *and suppose that* Γ *has the following properties:*

1. *if* (α,β) *and* (β,γ) *belong to* Γ*, then* (α,γ) *belongs to* Γ*;*
2. *if* (α,β) *belong to* Γ*, then every open interval in* (α,β) *belongs to* Γ*;*
3. *if* (α,β) *belong to* Γ *for every interval in* $[\alpha,\beta]\subset(c,d)$*, then* (c,d) *belongs to* Γ*;*
4. *if all of the intervals contiguous to the perfect set* $E\subset[a,b]$ *belong to* Γ*, then there exists an interval* I *in* Γ *such that* $I\cap E\neq\emptyset$*.*

Then Γ *contains the interval* (a,b)*.*

Lemma 3.6. *Suppose that* $f_n:[a,b]\to\mathbb{R}$*,* $f:[a,b]\to\mathbb{R}$*, and*

1. $f_n\to f$ *almost everywhere on* $[a,b]$ *as* $n\to\infty$*, where each* f_n *is Henstock (or* D^**) integrable on* $[a,b]$*;*
2. *the primitives* F_n *of* f_n *are continuous uniformly in* n *and* ACG^* *uniformly in* n*.*

Then f *is Henstock (or* D^**) integrable on* $[a,b]$ *and*

$$\lim_{n\to\infty}\int_a^b f_n=\int_a^b f. \tag{3.11}$$

Definition 3.7. Let $F:[a,b]\to X$ and let E be a subset of $[a,b]$.

(a) F is BV^* on E if $\sup\{\sum_i\omega(F;[c_i,d_i])\}$ is finite, where the supremum is taken over all finite collections $\{[c_i,d_i]\}$ of nonoverlapping intervals that have endpoints in E, ω denotes the oscillation of F over $[c_i,d_i]$, that is,

$$\omega(F;[c_i,d_i])=\sup\{\|F(x)-F(y)\|;\ x,y\in[c_i,d_i]\}. \tag{3.12}$$

(b) F is AC^* on E if for each $\epsilon>0$ there exists $\delta>0$ such that $\sum_i\omega(F;[c_i,d_i])<\epsilon$ whenever $\{[c_i,d_i]\}$ is a finite collection of nonoverlapping intervals that have endpoints in E and satisfy $\sum_i(d_i-c_i)<\delta$.

(c) F is BVG^* on E if E can be expressed as a countable union of sets on each of which F is BV^*.

(d) F is ACG^* on E if F is continuous on E and if E can be expressed as a countable union of sets on each of which F is AC^*.

Theorem 3.8. *Suppose that* X *is weakly sequentially complete and*

1. $f_n\to f$ *weakly almost everywhere on* $[a,b]$ *as* $n\to\infty$*, where each* f_n *is Henstock-Pettis integrable on* $[a,b]$*;*
2. *the primitives* F_n *of* f_n *are continuous uniformly in* n *and* ACG^* *uniformly in* n*.*

Then f is Henstock-Pettis integrable on $[a,b]$ and

$$\lim_{n\to\infty}\int_a^b f_n = \int_a^b f \text{ weakly.} \tag{3.13}$$

Proof. Let

$$\Gamma = \left\{(\alpha,\beta)\subset[a,b] : f \text{ is Henstock-Pettis integrable on } [\alpha,\beta],\ \int_\alpha^\beta f_n \longrightarrow \int_\alpha^\beta f \text{ weakly}\right\}. \tag{3.14}$$

We must show that Γ contains (a,b) and by Lemma 3.5 it is sufficient to verify that Γ satisfies Romanovski's four conditions.

Conditions (1) and (2) are easily verified.

Suppose that (α,β) belongs to Γ for every interval $[\alpha,\beta]$ in (c,d). For each positive integer $n>2/(d-c)$, define $I_n=(c+1/n,d-1/n)$ and let $x_n=x_{I_n}^{**}$.

Then we have

$$x_{(c,d)}^{**}(x^*) = \int_c^d x^*f = \lim_{n\to\infty}\int_{I_n} x^*f = \lim_{n\to\infty} x^*(x_n) \tag{3.15}$$

for each x^* in X^*. Since X is weakly sequentially complete, the sequence $\{x_n\}$ converges weakly to an element x_0 of X and we must have $x_{(c,d)}^{**}=x_0$. It follows easily that (c,d) belongs to Γ and this verifies condition (3).

Now let E be a perfect set in $[a,b]$ such that each of the intervals in $[a,b]$ contiguous to E belongs to Γ.

Since $\{F_n\}$ is continuous uniformly in n and ACG^* uniformly in n, then for each $x^*\in X^*$, $\{x^*F_n\}$ is continuous uniformly in n and ACG^* uniformly in n, and $x^*f_n\to x^*f$ almost everywhere in $[a,b]$. It follows from [1] that x^*f is special Denjoy integrable on $[a,b]$. So there exists an interval $[u,v]$ with $u,v\in E$ and $E\cap(u,v)\neq\emptyset$ such that $\{F_n\}$ is AC^* uniformly in n on $P=E\cap(u,v)$ and the series $\sum_k\omega(F_n;[u_k,v_k])$ unconditionally converges where $(u,v)-E=\cup_k(u_k,v_k)$. Hence $\sum_k\omega(\int_{u_k}^t x^*f_n;[u_k,v_k])<\infty$ for each $x^*\in X^*$. By Corollary 3.3, we have

$$\int_u^v f_n = \int_P f_n + \sum_k\int_{u_k}^{v_k} f_n. \tag{3.16}$$

$\{F_n\}$ is AC^* uniformly in n on P, $\{x^*F_n : x^*\in B(X^*),\ n\in\mathbb{N}\}$ is AC^* uniformly in n on P. So $\{x^*f_n : x^*\in B(X^*),\ n\in\mathbb{N}\}$ is uniformly integrable on P (see [2]), that is, for $E\subset P$,

$$\lim_{|E|\to 0}\int_E |x^*f_n| = 0 \quad \text{uniformly in } x^*\in B(X^*) \text{ and } n. \tag{3.17}$$

It follows from [4, Theorem 3] that f is Pettis integrable on P and $\int_P f_n\to\int_P f$ weakly. Since $\{F_n\}$ is AC^* uniformly in n on P, so for every $\epsilon>0$ there exists N such that $\sum_{k=N}^\infty\|\int_{u_k}^{v_k} f_n\|<\epsilon$, $n=1,2,\dots$. For every $x^*\in B(X^*)$, we have $\sum_{k=N}^\infty|\int_{u_k}^{v_k} x^*f_n|<\epsilon$, $n=1,2,\dots$. So $\sum_{k=N}^\infty|\int_{u_k}^{v_k} x^*f|<\epsilon$. Since X is weakly sequentially complete and X does not contain c_0, hence $\sum_k\int_{u_k}^{v_k} f$ unconditionally converges. By (3.16),

$$x^*\int_u^v f_n = x^*\int_P f_n + x^*\sum_k\int_{u_k}^{v_k} f_n. \tag{3.18}$$

Let $n \to \infty$, we have

$$x^{**}_{(u,v)}(x^*) = x^* \int_P f + x^* \sum_k \int_{u_k}^{v_k} f. \tag{3.19}$$

Hence

$$x^{**}_{(u,v)} = \int_P f + \sum_k \int_{u_k}^{v_k} f \in X, \tag{3.20}$$

that is, f is Henstock-Pettis integrable on $[u,v]$. So $(u,v) \in \Gamma$. This shows that (u,v) belongs to Γ and Γ satisfies condition (4). This completes the proof. □

Theorem 3.9. *Suppose that X is weakly sequentially complete and $f_n \to f$ weakly almost everywhere on $[a,b]$ as $n \to \infty$, where each f_n is Henstock-Pettis integrable on $[a,b]$. If there is a scalar function g with $\| f_n(\cdot) \| \le g(\cdot)$ almost everywhere for all n and if $\int g < \infty$, then f is Henstock-Pettis integrable on $[a,b]$ and*

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f \text{ weakly.} \tag{3.21}$$

Proof. It is similar to the proof of Theorem 3.8. □

Definition 3.10. Let $\{f_\alpha\}$ be a family of Henstock-Pettis integrable functions defined on $[a,b]$. The family $\{x^* f_\alpha : x^* \in B(X^*)\}$ is uniformly integrable in the generalized sense on $[a,b]$, if for each perfect set $E \subset [a,b]$ there exists an interval $[c,d] \subset [a,b]$ with $c,d \in E$ and $E \cap (c,d) \neq \emptyset$ such that $\{x^* f_\alpha : x^* \in B(X^*)\}$ is uniformly integrable on $P = E \cap (c,d)$ and for every α the series $\sum_k \int_{c_k}^{d_k} f_\alpha$ is unconditionally convergent where $(c,d) - E = \cup_k (c_k, d_k)$.

Theorem 3.11. *Suppose that X is weakly sequentially complete and*

(1) *$f_n \to f$ weakly almost everywhere on $[a,b]$ as $n \to \infty$, where each f_n is Henstock-Pettis integrable on $[a,b]$.*

(2) *The family $\{x^* f_n : x^* \in B(X^*),\ n \in \mathbb{N}\}$ is uniformly integrable in the generalized sense on $[a,b]$.*

(3) *For each $x^* \in X^*$, $\lim_{n\to\infty} \int_c^d x^* f_n = \int_c^d x^* f$ uniformly for every $[c,d] \subset [a,b]$.*

Then f is Henstock-Pettis integrable on $[a,b]$ and

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f \text{ weakly.} \tag{3.22}$$

Proof. It is similar to the proof of Theorem 3.8. The only difference is that the family $\{x^* f_n : x^* \in B(X^*),\ n \in \mathbb{N}\}$ is uniformly integrable in the generalized sense on $[a,b]$, then there is a portion $P = E \cap I$ of E such that the family $|x^* f_n \chi_E|$ is uniformly integrable on P. So f is Pettis integrable on P. □

Theorem 3.12. *Suppose that X is weakly sequentially complete and*

1. *$f_n \to f$ weakly almost everywhere on $[a,b]$ as $n \to \infty$, where each f_n is Henstock-Pettis integrable on $[a,b]$ and f is measurable,*
2. *the primitives F_n of f_n are weakly continuous uniformly in n and weakly ACG^* uniformly in n, that is, for every $x^* \in X^*$, $x^* F_n$ are continuous uniformly in n and ACG^* uniformly in n.*

Then f is Henstock-Pettis integrable on $[a,b]$ and

$$\lim_{n\to\infty}\int_a^b f_n = \int_a^b f \text{ weakly.} \tag{3.23}$$

Proof. For each x^* in X^*, we have

(1) $x^*f_n \to x^*f$ almost everywhere on $[a,b]$ as $n\to\infty$, where each x^*f_n is Henstock integrable on $[a,b]$,

(2) the primitives x^*F_n of x^*f_n are continuous uniformly in n and ACG^* uniformly in n. It follows from Lemma 3.6 that x^*f is Henstock integrable on $[a,b]$ and

$$\int_a^b x^*f_n \longrightarrow \int_a^b x^*f \quad \text{as } n\longrightarrow\infty. \tag{3.24}$$

By Theorem 2.2, f is Henstock-Dunford integrable on $[a,b]$. Since X is weakly sequentially complete and f is measurable, by Theorem 3.4, f is Henstock-Pettis integrable on $[a,b]$. □

Theorem 3.13. *Suppose that the unit ball $B(X^*)$ of X^* is weak* sequentially compact and*

(1) *$f_n \to f$ weakly almost everywhere in $[a,b]$ as $n\to\infty$, where each f_n is Henstock-Pettis integrable on $[a,b]$,*

(2) *the primitives F_n of f_n are continuous uniformly in n and ACG^* uniformly in n.*

Then f is Henstock-Pettis integrable on $[a,b]$ and

$$\lim_{n\to\infty}\int_a^b f_n = \int_a^b f \text{ weakly.} \tag{3.25}$$

Proof. Suppose that $I\subset I_0$. Let C be the weak closure of $\{\int_I f_n : n\in\mathbb{N}\}$. For each x^* in X^*, $\{x^*F_n : n\in\mathbb{N}\}$ is continuous uniformly in n and ACG^* uniformly in n in $[a,b]$, and further $\int_a^b x^*f_n = x^*\int_a^b f_n$. A convergence theorem, namely Lemma 3.6, guarantees that x^*f is Henstock integrable on $[a,b]$ and $\lim_{n\to\infty}\int_a^b x^*f_n = \int_a^b x^*f$ for each x^* in X^*. We observe that C is bounded and that $C-\{\int_I f_n : n\in\mathbb{N}\}$ contains at most one point. We will prove that C is weakly compact.

Suppose that C is not weakly compact. An appeal to a theorem of James [6, Theorem 1] produces a bounded sequence (x_k^*) in X^*, a sequence (x_n) in C, and an $\epsilon>0$ such that $x_k^*(x_n)=0$ for $k>n$ and $x_k^*(x_n)>\epsilon$ for $n\ge k$. By passing to subsequences and relabelling, we can find a subsequence $(\int_I g_n)$ of $(\int_I f_n)$ and a subsequence (y_k^*) of x_k^* such that

$$\begin{aligned} y_k^*\int_I g_n &= \int_I y_k^* g_n = 0 \quad \text{for } k>n,\\ y_k^*\int_I g_n &= \int_I y_k^* g_n > \epsilon \quad \text{for } n\ge k,\\ \lim_{n\to\infty}\int_I x^* g_n &= \int_I x^* f \quad \forall x^* \text{ in } X^*. \end{aligned} \tag{3.26}$$

Since the unit ball $B(X^*)$ of X^* is weak* sequentially compact, the sequence (y_k^*) has a subsequence $(y_{k_j}^*)$ which weak* converges to y_0^*, so $\lim_{j\to\infty} y_{k_j}^* f = y_0^* f$ on I_0,

$\lim_{j\to\infty} y_{k_j}^* F = y_0^* F$ on I_0, that is, $\lim_{j\to\infty}\int_I y_{k_j}^* f = \int_I y_0^* f$. To force a contradiction, note that for each k, $\lim_{n\to\infty}\int_I y_k^* f_n = \int_I y_k^* f$. Hence $\int_I y_k^* f \geq \epsilon$ for each k, and $\int_I y_0^* f \geq \epsilon$. On the other hand, notice that since each g_n is Henstock-Pettis integrable, $(y_{k_j}^*)$ weak* converges to y_0^*, hence

$$\lim_{j\to\infty}\int_I y_{k_j}^* g_n = \lim_{j\to\infty} y_{k_j}^* \int_I g_n = y_0^* \int_I g_n = \int_I y_0^* g_n. \tag{3.27}$$

Since this holds for each n, and since $\lim_{n\to\infty}\int_I y_0^* g_n = \int_I y_0^* f$, we see that $\int_I y_0^* f = 0$. This contradicts the inequality $\int_I y_0^* f \geq \epsilon$, and proves that C is weakly compact. Since $\lim_{n\to\infty}\int_I x^* f_n = \int_I x^* f$, the sequence $(\int_I f_n)$ of the Henstock-Pettis integrals is weakly Cauchy. It follows from the weak compactness of C that $\lim_{n\to\infty}\int_I f_n$ exists weakly in X. Denote $F(I) = \int_I f = \lim_{n\to\infty}\int_I f_n$ weakly, then $x^*F(I) = x^*\int_I f = \int_I x^* f$ for each x^* in X^*. So f is Henstock-Pettis integrable on $[a,b]$ and

$$\lim_{n\to\infty}\int_a^b f_n = \int_a^b f \text{ weakly.} \tag{3.28}$$

□

Corollary 3.14. *Suppose that X is a reflexive Banach space and*

(1) *$f_n \to f$ weakly almost everywhere on $[a,b]$ as $n\to\infty$, where each f_n is Henstock-Pettis integrable on $[a,b]$,*
(2) *the primitives F_n of f_n are weakly continuous uniformly in n and weakly ACG^* uniformly in n on $[a,b]$.*

Then f is Henstock-Pettis integrable on $[a,b]$ and

$$\lim_{n\to\infty}\int_a^b f_n = \int_a^b f \text{ weakly.} \tag{3.29}$$

Theorem 3.15. *If the following conditions are satisfied:*

(1) *$\lim_{n\to\infty} f_n = f$ weakly almost everywhere on $[a,b]$, where each f_n is Henstock-Dunford integrable on $[a,b]$,*
(2) *the primitives F_n of f_n are weakly continuous uniformly in n and weakly ACG^* uniformly in n.*

Then f is Henstock-Dunford integrable on $[a,b]$ and

$$\lim_{n\to\infty}\int_a^b f_n = \int_a^b f \text{ weakly.} \tag{3.30}$$

Proof. Since

(1) $\lim_{n\to\infty} x^* f_n = x^* f$ almost everywhere on $[a,b]$,
(2) the primitives x^*F_n of x^*f_n are continuous uniformly in n and ACG^* uniformly in n.

Then, as in the proof of Theorem 3.12, x^*f is Henstock integrable on $[a,b]$ and

$$\lim_{n\to\infty}\int_a^b x^* f_n = \int_a^b x^* f. \tag{3.31}$$

By Theorem 2.2, f is Henstock-Dunford integrable on $[a,b]$ and

$$\lim_{n\to\infty}\int_a^b f_n = \int_a^b f \text{ weakly.} \tag{3.32}$$

□

References

[1] V. G. Čelidze and A. G. Džvaršeĭšvili, *The Theory of the Denjoy Integral and some Applications*, Series in Real Analysis, vol. 3, World Scientific Publishing Co., New Jersey, 1989, translated from Russian, with a preface and an appendix by P. S. Bullen. MR 90k:26013. Zbl 744.26006.

[2] J. Diestel and J. J. Uhl, Jr., *Vector Measures*, Mathematical Surveys, no. 15, American Mathematical Society, Rhode Island, 1977, with a foreword by B. J. Pettis. MR 56#12216. Zbl 369.46039.

[3] J. L. Gámez and J. Mendoza, *On Denjoy-Dunford and Denjoy-Pettis integrals*, Studia Math. **130** (1998), no. 2, 115–133. MR 99k:28013. Zbl 980.39999.

[4] R. F. Geitz, *Pettis integration*, Proc. Amer. Math. Soc. **82** (1981), no. 1, 81–86. MR 82c:28018. Zbl 506.28007.

[5] R. A. Gordon, *The Denjoy extension of the Bochner, Pettis, and Dunford integrals*, Studia Math. **92** (1989), no. 1, 73–91. MR 90b:28011. Zbl 681.28006.

[6] R. C. James, *Weak compactness and reflexivity*, Israel J. Math. **2** (1964), 101–119. MR 31#585. Zbl 127.32502.

[7] P. Y. Lee, *Lanzhou Lectures on Henstock Integration*, Series in Real Analysis, vol. 2, World Scientific Publishing Co., New Jersey, 1989. MR 92j:26010. Zbl 699.26004.

[8] G. Q. Liu, *On necessary conditions for Henstock integrability*, Real Anal. Exchange **18** (1992/93), no. 2, 522–531. MR 94e:26014. Zbl 788.26008.

Ye Guoju: Department of Mathematics, Northwest Normal University, Lanzhou 730070, China

E-mail address: yeguoju@21cn.com

An Tianqing: Department of Mathematics, Northwest Normal University, Lanzhou 730070, China