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A GAUSS TYPE FUNCTIONAL EQUATION

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ABSTRACT. Gauss' functional equation (used in the study of the arithmetic-geometric mean) is generalized by replacing the arithmetic mean and the geometric mean by two arbitrary means.

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1. Introduction. By mean we understand a function $M : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ which satisfies the condition

$$\min(a,b) \le M(a,b) \le \max(a,b) \quad \forall a,b > 0. \tag{1.1}$$

The mean is called symmetric if

$$M(a,b) = M(b,a) \quad \forall a,b > 0. \tag{1.2}$$

Usual examples are the power means given by

$$P_n(a,b) = \left(\frac{a^n + b^n}{2}\right)^{1/n} \tag{1.3}$$

for $n \ne 0$, while for $n = 0$ it is the geometric mean

$$P_0(a,b) = G(a,b) = \sqrt{ab}. \tag{1.4}$$

Of course, the arithmetic mean is $A = P_1$.

If M is a mean and $p : \mathbb{R}_+ \to \mathbb{R}$ is a strictly monotonous function, the expression

$$M(p)(a,b) = p^{-1}[M(p(a),p(b))] \tag{1.5}$$

defines another mean $M(p)$ which is called M-quasi mean (see [1]). For example, the power means are A-quasi means. More exactly $P_n = A(e_n)$, where

$$e_n(x) = x^n \quad \text{for } n \ne 0, \qquad e_0(x) = \ln x. \tag{1.6}$$

In what follows, we refer to another famous example of mean. Given two positive numbers a and b, we define define successively the terms

$$a_{n+1} = A(a_n,b_n), \quad b_{n+1} = G(a_n,b_n), \quad n \ge 0, \tag{1.7}$$

where $a_o = a$ and $b_o = b$. It is known (see [1]) that $(a_n)_{n\ge o}$ and $(b_n)_{n\ge o}$ are convergent to a common limit which is denoted by $A \otimes G(a,b)$. It defines the arithmetic-geometric mean of Gauss $A \otimes G$.

The following representation formula is also known (see [1])

$$A \otimes G(a,b) = [f(a,b)]^{-1}, \tag{1.8}$$

where

$$f(a,b) = \frac{1}{2\cdot\pi}\int_0^{2\pi} \frac{d\theta}{\sqrt{a^2\cdot\cos^2\theta + b^2\cdot\sin^2\theta}}. \tag{1.9}$$

The proof of this formula is based on the fact that the function f verifies the relation

$$f(A(a,b),G(a,b)) = f(a,b), \tag{1.10}$$

which is called Gauss' functional equation.

These results were generalized as follows. We denote

$$\begin{gathered} r_n(\theta) = (a^n\cdot\cos^2\theta + b^n\cdot\sin^2\theta)^{1/n}, \quad n \neq 0, \\ r_0(\theta) = \lim_{n\to 0} r_n(\theta) = a^{\cos^2\theta} b^{\sin^2\theta}. \end{gathered} \tag{1.11}$$

If $p : \mathbb{R}_+ \to \mathbb{R}$ is a strictly monotonous function, then

$$M_{p,n}(a,b) = p^{-1}\left(\frac{1}{2\pi}\int_0^{2\pi} p(r_n(\theta))\,d\theta\right) \tag{1.12}$$

defines a symmetric mean. The arithmetic-geometric mean of Gauss is obtained for $n = 2$ and $p(x) = x^{-1}$. For $n = -2$ and $p(x) = x^{-2}$ the mean can be found in [7]. The case $n = 1$ and $p = \log$ was studied in [2]. The essential step was done in [4] by the consideration of the definition (1.12) for $n = 2$ with an arbitrary p. The values $n = -1$ and $n = 1$ were studied in [5, 6]. The general case (of arbitrary n) was studied in [8] and continued in [9]. In [8], Gauss' functional equation was also replaced by a more general equation

$$F(P_q(a,b),P_s(a,b)) = F(a,b). \tag{1.13}$$

In this paper, we generalize the mean (1.12) as well as the functional equation (1.13).

2. An integral mean. We consider the strictly monotonous functions p and q. Using them, we define the functions

$$\begin{gathered} r_q(\theta) = q^{-1}[q(a)\cdot\cos^2\theta + q(b)\cdot\sin^2\theta], \\ f(a,b;p,q) = \frac{1}{2\pi}\int_0^{2\pi} p[r_q(\theta)]\,d\theta. \end{gathered} \tag{2.1}$$

It is easy to prove that

$$M_{p,q}(a,b) = p^{-1}[f(a,b;p,q)] \tag{2.2}$$

defines a mean. Choosing $q = e_n$ we obtain $M_{p,q} = M_{p,n}$. We so have generalized the means (1.12). On the other hand, if we let $p\circ q^{-1} = Q$, we have $M_{p,q} = M_{Q,1}(q)$. Thus, $M_{p,q}$ is a $M_{Q,1}$-quasi mean.

It is thus enough to consider the function

$$f(a,b;p) = \frac{1}{2\pi}\int_0^{2\pi} p(a\cdot\cos^2\theta + b\cdot\sin^2\theta)\,d\theta \tag{2.3}$$

which defines the mean $M_p = M_{p,1}$ by

$$M_p(a,b) = p^{-1}[f(a,b;p)]. \tag{2.4}$$

In what follows, we assume that the function p is two times differentiable. From any of the papers [5, 6, 8, 9], we can deduce the following result.

Lemma 2.1. *The function f defined by (2.3) has the following partial derivatives:*

$$\begin{aligned} f'_a(c,c;p) &= f'_b(c,c;p) = \frac{1}{2}\cdot p'(c), \\ f''_{aa}(c,c;p) &= f''_{bb}(c,c;p) = \frac{3}{8}\cdot p''(c), \\ f''_{ab}(c,c;p) &= \frac{1}{8}\cdot p''(c). \end{aligned} \tag{2.5}$$

3. The functional equation. We replace (1.13) by a more general functional equation

$$F(M(a,b),N(a,b)) = F(a,b), \tag{3.1}$$

where M and N are two given means. We prove the following result.

Lemma 3.1. *If the function f, defined by (2.3), verifies the functional equation (3.1), then the function p is a solution of the differential equation*

$$\begin{aligned} p''(c)\cdot\Big\{&[3\cdot M'_b(c,c)+N'_b(c,c)]\cdot M'_a(c,c)+[M'_b(c,c)+3\cdot N'_b(c,c)]\cdot N'_a(c,c)-1\Big\} \\ &+4\cdot p''(c)\cdot[M''_{ab}(c,c)+N''_{ab}(c,c)] = 0. \end{aligned} \tag{3.2}$$

Proof. Taking in (3.1) the partial derivatives with respect to a, we obtain

$$F'_a[M(a,b),N(a,b)]\cdot M'_a(a,b)+F'_b[M(a,b),N(a,b)]\cdot N'_a(a,b) = F'_a(a,b). \tag{3.3}$$

Taking again the derivatives with respect to b, it follows that

$$\begin{aligned} &\Big\{F''_{aa}[M(a,b),N(a,b)]\cdot M'_b(a,b)+F''_{ab}[M(a,b),N(a,b)]\cdot N'_b(a,b)\Big\}\cdot M'_a(a,b) \\ &\quad+\Big\{F''_{ab}[M(a,b),N(a,b)]\cdot M'_b(a,b)+F''_{bb}[M(a,b),N(a,b)]\cdot N'_b(a,b)\Big\}\cdot N'_a(a,b) \\ &\quad+F'_a[M(a,b),N(a,b)]\cdot M''_{ab}(a,b)+F'_b[M(a,b),N(a,b)]\cdot N''_{ab}(a,b) = F''_{ab}(a,b). \end{aligned} \tag{3.4}$$

For $a = b = c$ and the function $F = f$, defined by (2.3), we apply Lemma 2.1 and obtain (3.2). □

Consequence 3.2. If the function f, defined by (2.3), verifies the functional equation (3.1), where the means M and N are symmetric, the function p is a solution of the differential equation

$$p''(c)+4\cdot p'(c)\cdot[M''_{ab}(c,c)+N''_{ab}(c,c)] = 0. \tag{3.5}$$

Proof. As the means are symmetric, their partial derivatives of the first order are equal to $1/2$ (see [3]), thus (3.2) becomes (3.5). □

Consequence 3.3. If the function f, defined by (2.3), verifies the functional equation (1.13), then the function p is given by

$$p(c) = C \cdot c^{r+s-1} + D, \tag{3.6}$$

where C and D are arbitrary constants.

Proof. We have in (3.5), $M = P_r$ and $N = P_s$. Thus

$$M''_{ab}(c,c) = \frac{1-r}{4 \cdot c}, \qquad N''_{ab}(c,c) = \frac{1-s}{4 \cdot c}. \tag{3.7}$$

Replacing them in (3.5), we obtain the differential equation

$$p''(c) + \frac{2-r-s}{c} \cdot p'(c) = 0 \tag{3.8}$$

with the solution given above. □

Remark 3.4. This last result was obtained in [8]. As it is shown in [9], the condition is also sufficient for $r = -s = 1$.

Remark 3.5. Equation (3.1) can be further generalized at

$$F(g(M(a,b)), g(N(a,b))) = h(F(a,b)), \tag{3.9}$$

where g and h are two given functions. We have in view the following result given in [2]. The function f, defined by (2.3), verifies the relation

$$f(A^2(a,b), G^2(a,b); \log) = 2 \cdot f(a,b;\log). \tag{3.10}$$

4. Special means. A problem which is studied for the integral means defined in [4, 5, 6, 8, 9] is that of the determination of the cases in which the mean reduces at a given one, usually a power mean. Similar results can be given also in more general circumstances. We prove the following lemma.

Lemma 4.1. *If for a given mean N, we have $M_p = N$, then the function p verifies the equation*

$$p''(c) \cdot [8 \cdot N'_a(c,c) \cdot N'_b(c,c) - 1] + 8 \cdot p'(c) \cdot N''_{ab}(c,c) = 0. \tag{4.1}$$

Proof. In the given hypotheses, we have

$$f(a,b;p) = p[N(a,b)]. \tag{4.2}$$

Taking the partial derivatives with respect to a, we have

$$f'_a(a,b;p) = p'[N(a,b)] \cdot N'_a(a,b). \tag{4.3}$$

Then we take the derivative with respect to b, we obtain

$$f''_{ab}(a,b;p) = p''[N(a,b)] \cdot N'_a(a,b) \cdot N'_b(a,b) + p'[N(a,b)] \cdot N''_{ab}(a,b). \tag{4.4}$$

For $a = b = c$, Lemma 2.1 gives (4.1). □

Consequence 4.2. If we have $M_p = N$, with the symmetric mean N, then the function p verifies the equation

$$p''(c) + 8 \cdot p'(c) \cdot N''_{ab}(c,c) = 0. \tag{4.5}$$

Consequence 4.3. If we have $M_p = P_r$, then the function p is given by

$$p(c) = C \cdot c^{2r-1} + D, \tag{4.6}$$

where C and D are arbitrary constants.

Remark 4.4. In [9], it is shown that the above condition is also sufficient for $r = 0$, $1/2$, and 1.

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