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ON A NONRESONANCE CONDITION BETWEEN THE FIRST AND THE SECOND EIGENVALUES FOR THE p-LAPLACIAN

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ABSTRACT. We are concerned with the existence of solution for the Dirichlet problem $-\triangle_p u = f(x,u) + h(x)$ in Ω, $u = 0$ on $\partial\Omega$, when $f(x,u)$ lies in some sense between the first and the second eigenvalues of the p-Laplacian $\triangle_p$. Extensions to more general operators which are $(p-1)$-homogeneous at infinity are also considered.

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1. Introduction. In this paper, we are concerned with the existence of solution to the following quasilinear elliptic problem:

$$\begin{gathered} -\triangle_p u = f(x,u) + h(x) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.1}$$

Here Ω is a smooth bounded domain of $\mathbb{R}^N$, $N \geq 1$, Δ_p denotes the p-Laplacian $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $1 < p < \infty$, h belongs to $W^{-1,p'}(\Omega)$ with p' the Hölder conjugate of p and f is a Caratheodory function from $\Omega \times \mathbb{R}$ to $\mathbb{R}$ such that

$$\lambda_1 \underset{\neq}{\leq} \liminf_{s\to\pm\infty} \frac{f(x,s)}{|s|^{p-2}s} \leq \limsup_{s\to\pm\infty} \frac{f(x,s)}{|s|^{p-2}s} < \lambda_2 \quad \text{a.e. in } \Omega, \tag{1.2}$$

where λ_1 (resp., λ_2) is the first (resp., the second) eigenvalue of the problem

$$\begin{gathered} -\Delta_p u = \lambda |u|^{p-2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.3}$$

Problems of this sort have been extensively studied in the 70s and 80s in the semilinear case $p = 2$. In the quasilinear case $p \neq 2$, (1.1) was investigated for $N = 1$ in [6] and for $N \geq 1$ in [3]. In this latter work nonresonance is studied at the left of λ_1.

One of the difficulties to deal with the partial differential equation case $N \geq 1$ is the lack of knowledge of the spectrum of the p-Laplacian in that case. The basic properties of λ_1 were established in [2], while a variational characterization of λ_2 was derived recently in [4]. This variational characterization of λ_2 allows the study of its (strict) monotonicity dependence with respect to a weight. This is the property which is used in our approach to (1.1). The asymmetry in our assumption (1.2) between λ_1 and λ_2 also comes from that property. In fact it remains an open question whether the last strict inequality in (1.2) can be replaced by $\underset{\neq}{\leq}$.

In Section 3 we extend our existence result to more general operators. We consider

$$A(u) = f(x,u) + h(x) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.4}$$

where $A = -\sum_{i=1}^{N}(\partial/\partial x_i)A_i(x,u,\nabla u)$ verifies a $(p-1)$-homogeneity condition at infinity. Such operators were studied by Anane [1] in the variational case. Here we use degree theory for mappings of type $(S)_+$ as developed by Browder [7] and Berkowits and Mustonen [5]. No variational structure is consequently needed.

2. A result for the p-Laplacian.

We seek a weak solution of (1.1), that is,

$$\begin{gathered} \text{find } u \in W_0^{1,p}(\Omega) \quad \text{such that} \quad \forall v \in W_0^{1,p}(\Omega): \\ \int_\Omega |\nabla u|^{p-2}\nabla u.\nabla v\,dx = \int_\Omega f(x,u)v\,dx + \langle h, v\rangle, \end{gathered} \tag{2.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality product between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$. We assume that f satisfies

$$\max_{|s|\le R}|f(x,s)| \in L^{p'}(\Omega), \quad \forall R > 0, \tag{2.2}$$

$$\lambda_1 \underset{\not\equiv}{\le} l(x) \le k(x) < \lambda_2 \quad \text{a.e. in } \Omega, \tag{2.3}$$

where

$$l(x) = \liminf_{s\to\pm\infty}\frac{f(x,s)}{|s|^{p-2}s}, \qquad k(x) = \limsup_{s\to\pm\infty}\frac{f(x,s)}{|s|^{p-2}s}. \tag{2.4}$$

The first inequality in (2.3) must be understood as "less or equal almost everywhere together with strict inequality on a set of positive measure." We also assume that some uniformity holds in the inequalities in (2.3):

$$\begin{aligned} &\forall\varepsilon > 0, \quad \exists\eta(\varepsilon) > 0 : \lambda_1 - \varepsilon \le \frac{f(x,s)}{|s|^{p-2}s}, \quad \forall|s| \ge \eta(\varepsilon), \quad \text{a.e. in } \Omega, \\ &\forall\varepsilon > 0, \quad \exists\eta(\varepsilon) > 0 : \frac{f(x,s)}{|s|^{p-2}s} \le \lambda_2 + \varepsilon, \quad \forall|s| \ge \eta(\varepsilon), \quad \text{a.e. in } \Omega. \end{aligned} \tag{2.5}$$

Remark 2.1. It is clear that (2.2) and (2.5) imply the growth condition

$$|f(x,s)| \le a|s|^{p-1} + b(x) \quad \forall s \in \mathbb{R}, \text{ a.e. in } \Omega, \tag{2.6}$$

where $a > 0$ and $b(\cdot) \in L^{p'}(\Omega)$.

Remark 2.2. Equations (2.2) and (2.5) also imply

$$\begin{gathered} \forall\varepsilon > 0, \quad \exists b_\varepsilon \in L^{p'}(\Omega) \text{ such that} \\ |s|^p(\lambda_1 - \varepsilon) - b_\varepsilon(x) \le sf(x,s) \le |s|^p(\lambda_2 + \varepsilon) + b_\varepsilon(x), \\ \forall s \in \mathbb{R}, \quad \text{a.e. in } \Omega. \end{gathered} \tag{2.7}$$

Theorem 2.3. *Suppose that f satisfies (2.2), (2.3), and (2.5). Then for any $h \in W^{-1,p'}(\Omega)$, problem (2.1) admits a solution u in $W_0^{1,p}(\Omega)$.*

Proof. We denote by $(T_t)_{t\in[0,1]}$ the family of operators from $W_0^{1,p}(\Omega)$ to $W_0^{1,p}(\Omega)$ defined by

$$T_t(u) = (-\Delta_p)^{-1}[(1-t)\alpha|u|^{p-2}u + tf(\cdot,u) + th(\cdot)], \tag{2.8}$$

where α is some fixed number with $\lambda_1 < \alpha < \lambda_2$.

To prove Theorem 2.3, we first establish the following estimate:

$$\exists R > 0 \text{ such that } \forall t \in [0,1],\ \forall u \in \partial B(O,R) \text{ such that } [I - T_t](u) \neq 0, \tag{2.9}$$

where $B(O,R)$ denotes the ball of center O and radius R in $W_0^{1,p}(\Omega)$.

To prove (2.9) we assume by contradiction that

$$\forall n > 0, \quad \exists t_n \in [0,1], \exists u_n \in W_0^{1,p}(\Omega) \text{ with } ||u_n||_{1,p} = n \text{ such that } T_{t_n}(u_n) = u_n, \tag{2.10}$$

where $\|\cdot\|_{1,p}$ denotes the norm in $W_0^{1,p}(\Omega)$.

Let $w_n = u_n/n$. We can extract from (w_n) a subsequence, still denoted by (w_n), which converges weakly in $W_0^{1,p}(\Omega)$, strongly in $L^p(\Omega)$ and a.e. in Ω to $w \in W_0^{1,p}(\Omega)$. We can also suppose that t_n converges to $t \in [0,1]$. To reach a contradiction, we use the following lemmas which give various information on w_n and w.

Lemma 2.4. *The sequence g_n defined by*

$$g_n = \frac{f(x,nw_n)}{n^{p-1}} \tag{2.11}$$

is bounded in $L^{p'}(\Omega)$, and consequently, for a subsequence, g_n converges weakly to some g in $L^{p'}(\Omega)$.

Proof. This is an immediate consequence of (2.6). □

Lemma 2.5. *$w \not\equiv 0$.*

Proof. Since w_n verifies,

$$\begin{aligned}\int_\Omega |\nabla w_n|^p\,dx &= (1-t_n)\alpha\int_\Omega |w_n|^p\,dx \\ &\quad + t_n\left[\int_\Omega g_n(x)w_n(x)\,dx + \frac{1}{n^{p-1}}\langle h, w_n\rangle\right],\end{aligned} \tag{2.12}$$

we deduce from Lemma 2.4 that

$$1 = (1-t)\alpha\int_\Omega |w|^p\,dx + t\int_\Omega g(x)w(x)\,dx, \tag{2.13}$$

which clearly implies the conclusion of Lemma 2.5. □

Lemma 2.6. *$g = 0$ a.e. in $\Omega\setminus A$, where $A = \{x \in \Omega : w(x) \neq 0\}$.*

Proof. By (2.6), we have

$$|g_n(x)| \le a|w_n|^{p-1} + \frac{b(x)}{n^{p-1}} \quad \text{a.e. in } \Omega, \tag{2.14}$$

and so

$$\|g_n\|_{L^{p'}(\Omega\setminus A)} \le a\|w_n\|^{p/p'}_{L^p(\Omega\setminus A)} + \frac{1}{n^{p-1}}\|b\|_{L^{p'}(\Omega\setminus A)}, \tag{2.15}$$

which implies

$$\lim_{n\to+\infty}\|g_n\|_{L^{p'}(\Omega\setminus A)} = 0. \tag{2.16}$$

Set $D = \{x \in \Omega\setminus A : g(x) \ne 0\}$. By Lemma 2.4 we have, for $\phi(x) = \operatorname{sign}[g(x)]\chi_D(x) \in L^p(D)$

$$\lim_{n\to+\infty}\int_D g_n(x)\phi(x)\,dx = \int_D |g(x)|\,dx, \tag{2.17}$$

and consequently by (2.16),

$$\int_D |g(x)|\,dx = 0, \tag{2.18}$$

which implies $\operatorname{meas}(D) = 0$, that is, the conclusion of Lemma 2.6. □

Lemma 2.7. *Set*

$$\tilde{g}(x) = \begin{cases} \dfrac{g(x)}{|w(x)|^{p-2}w(x)} & \text{on } A, \\ \beta & \text{on } \Omega\setminus A, \end{cases} \tag{2.19}$$

where β is a fixed number with $\lambda_1 < \beta < \lambda_2$. We have

$$\lambda_1 \lneq \tilde{g}(x) < \lambda_2 \quad \text{a.e. in } \Omega. \tag{2.20}$$

Proof. Set

$$\begin{aligned} B_l &= \left\{x \in A : w(x)g(x) < l(x)|w(x)|^p\right\}, \\ B_k &= \left\{x \in A : w(x)g(x) > k(x)|w(x)|^p\right\}. \end{aligned} \tag{2.21}$$

We first prove that $\operatorname{meas}(B_l) = 0$ and $\operatorname{meas}(B_k) = 0$.

By (2.7), we have that $\forall \varepsilon \ge 0, \exists b_\varepsilon \in L^{p'}(\Omega)$ such that

$$\begin{aligned} &-\frac{b_\varepsilon(x)}{n^p} + |w_n(x)|^p[l(x)-\varepsilon] \\ &\qquad \le w_n(x)g_n(x) \le \frac{b_\varepsilon(x)}{n^p} + |w_n(x)|^p[k(x)+\varepsilon] \quad \text{a.e. in } \Omega. \end{aligned} \tag{2.22}$$

The first inequality gives

$$-\frac{1}{n^p}\int_{B_l} b_\varepsilon(x)\,dx + \int_{B_l}|w_n(x)|^p[l(x)-\varepsilon]\,dx \le \int_{B_l} w_n(x)g_n(x)\,dx. \tag{2.23}$$

Letting first $x \to \infty$, then $\varepsilon \to 0$, we deduce

$$\int_{B_l}\left[w(x)g(x) - |w(x)|^p l(x)\right]dx \ge 0, \tag{2.24}$$

which implies $\text{meas}(B_l) = 0$. Similarly one gets $\text{meas}(B_k) = 0$. We thus have

$$l(x) \le \tilde{g}(x) \le k(x) \quad \text{a.e. in } A. \tag{2.25}$$

Since

$$\lambda_1 < \tilde{g}(x) = \beta < \lambda_2 \quad \text{a.e. in } \Omega \setminus A, \tag{2.26}$$

we obtain the conclusion of the lemma. □

Lemma 2.8. *w is a solution of*

$$\begin{gathered} -\Delta_p w = m|w|^{p-2}w \quad \textit{in } \Omega, \\ w = 0 \quad \textit{on } \partial\Omega, \end{gathered} \tag{2.27}$$

where $m(x) = (1-t)\alpha + t\tilde{g}(x)$.

Proof. We first prove that w is a solution of

$$\begin{gathered} -\Delta_p w = (1-t)\alpha|w|^{p-2}w + tg \quad \text{in } \Omega, \\ w = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.28}$$

We recall that w_n satisfies

$$\begin{gathered} -\Delta_p w_n = (1-t_n)\alpha|w_n|^{p-2}w_n + t_n\left[g_n + \frac{1}{n^{p-1}}h\right] \quad \text{in } \Omega, \\ w_n = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.29}$$

Since $(-\Delta_p)(w_n)$ is bounded in $W^{-1,p'}(\Omega)$, there exists a subsequence, still denoted by (w_n), and a distribution $T \in W^{-1,p'}(\Omega)$, such that $(-\Delta_p)(w_n)$ converges weakly to T in $W^{-1,p'}(\Omega)$; in particular

$$\lim_{n\to+\infty} \langle -\Delta_p w_n, w\rangle = \langle T, w\rangle. \tag{2.30}$$

We also have

$$\begin{aligned} \langle -\Delta_p w_n, w_n - w\rangle &= (1-t_n)\alpha\int_\Omega |w_n|^{p-2}w_n(w_n-w)\,dx \\ &\quad + t_n\left[\int_\Omega g_n(x)(w_n-w)\,dx + \frac{1}{n^{p-1}}\langle h, w_n-w\rangle\right], \end{aligned} \tag{2.31}$$

which implies

$$\lim_{n\to+\infty} \langle -\Delta_p w_n, w_n - w\rangle = 0, \tag{2.32}$$

and therefore

$$\lim_{n\to+\infty} \langle -\Delta_p w_n, w_n\rangle = \langle T, w\rangle. \tag{2.33}$$

Since $(-\Delta_p)$ is an operator of type (M), we deduce

$$T = -\Delta_p w. \tag{2.34}$$

Going to the limit in (2.29) then yields (2.28). But by Lemma 2.6, we have

$$(1-t)\alpha|w|^{p-2}w + tg = m|w|^{p-2}w \quad \text{a.e. in } \Omega. \tag{2.35}$$

So w is a solution of (2.27). □

We denote by $\lambda_1(\Omega, r(x))$ (resp., $\lambda_2(\Omega, r(x))$) the first (resp., the second) eigenvalue in the problem with weight

$$\begin{gathered} -\Delta_p u = \lambda r(x)|u|^{p-2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.36}$$

By Lemma 2.7 and the fact that $\lambda_1 < \alpha < \lambda_2$, we have

$$\lambda_1 \lneqq m(x) < \lambda_2 \quad \text{a.e. in } \Omega. \tag{2.37}$$

It follows, by the strict monotonicity property of the second eigenvalue with respect to the weight (cf. [4]), that

$$1 = \lambda_2(\Omega, \lambda_2) < \lambda_2(\Omega, m). \tag{2.38}$$

It also follows by the strict monotonicity of the first eigenvalue with respect to the weight (cf. [8]), that

$$\lambda_1(\Omega, m) < \lambda_1(\Omega, \lambda_1) = 1. \tag{2.39}$$

Consequently,

$$\lambda_1(\Omega, m) < 1 < \lambda_2(\Omega, m). \tag{2.40}$$

But by Lemmas 2.5 and 2.8, 1 is an eigenvalue of $(-\Delta_p)$ for the weight m. This contradicts the definition of the second eigenvalue $\lambda_2(\Omega, m)$. We have thus proved that the estimate (2.9) holds.

We can now conclude by a standard degree argument. Indeed T_t is clearly completely continuous, since $(\Delta_p)^{-1}$ is continuous from $W^{-1,p'}(\Omega)$ to $W_0^{1,p}(\Omega)$. Therefore,

$$\deg\left(I - T_0, B(O,R), O\right) = \deg\left(I - T_1, B(O,R), O\right). \tag{2.41}$$

Since T_0 is odd, we have, by Borsuk theorem, that $\deg(I - T_0, B(O,R), O)$ is an odd integer and so nonzero. It then follows that there exists $u \in B(O,R)$ such that $T_1(u) = u$, which proves Theorem 2.3. □

3. Generalization.

Theorem 2.3 will now be extended to the case of nonhomogeneous operators. We consider the problem

$$\begin{gathered} A(u) = f(x,u) + h(x) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.1}$$

where

$$A(u) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} A_i(x, u(x), \nabla u(x)). \tag{3.2}$$

The method used in Section 2 for $(-\Delta_p)$ can be adapted under suitable assumptions on A. We basically assume that A is a Leray-Lions operator which is $(p-1)$-homogeneous at infinity. Our precise assumptions are the following:

$$\text{Each } A_i(x,s,\xi) \text{ is a Carathéodory function,} \tag{3.3}$$

$$\sum_{i=1}^{N}\left[A_i(x,s,\xi)-A_i\left(x,s,\xi'\right)\right](\xi_i-\xi_i')>0, \quad \text{for a.e. } x\in\Omega,\ \text{all } s\in\mathbb{R},\ \text{all } \xi\neq\xi'\in\mathbb{R}^N, \tag{3.4}$$

$$\exists K\in L^{p'}(\Omega), \exists c(t) \text{ a function defined on } \mathbb{R}^+ \text{ with } \lim_{t\to+\infty}c(t)=0 \text{ such that}$$
$$\left|A_i(x,ts,t\xi)-t^{p-1}|\xi|^{p-2}\xi_i\right|\le t^{p-1}c(t)\left[|\xi|^{p-1}+|s|^{p-1}+K(x)\right], \tag{3.5}$$
$$\text{for a.e. } x\in\Omega,\ \text{all } s\in\mathbb{R},\ \text{all } \xi\in\mathbb{R}^N,\ \text{all } t\in\mathbb{R}^+.$$

We will be able to solve (3.1) when $f(x,s)$ lies at infinity between the first and the second eigenvalues of the p-Laplacian $(-\Delta_p)$, in the sense of (1.2).

Remark 3.1. Equation (3.5) is a hypothesis which means that A is asymptotically homogeneous to $(-\Delta_p)$. An example of an operator which verifies (3.3), (3.4), and (3.5) is the following regularized version of the p-Laplacian:

$$A=-\Delta_{p,\epsilon}=-\operatorname{div}\left[(\epsilon+|\nabla u|^2)^{(p-2)/2}\nabla u\right] \tag{3.6}$$

with $\epsilon>0$.

Remark 3.2. Equations (3.3), (3.4), and (3.5) imply the following usual growth and coercivity conditions:

$$\begin{gathered}\exists c_4>0,\ \exists K_4\in L^{p'}(\Omega) \text{ such that } \left|A_i(x,s,\xi)\right|\le c_4\left(|\xi|^{p-1}+|s|^{p-1}+K_4(x)\right),\\ \text{a.e. } x\in\Omega,\ \forall s\in\mathbb{R},\ \xi\in\mathbb{R}^N,\ \text{for } i=1,\dots,N,\end{gathered} \tag{3.7}$$

$$\begin{gathered}\exists c_5>0,\ c_5'>0,\ K_5\in L^1(\Omega) \text{ such that } \sum_{i=1}^{N}A_i(x,s,\xi)\xi_i\ge c_5|\xi|^p-c_5'|s|^p-K_5(x),\\ \text{a.e. } x\in\Omega,\ \forall s\in\mathbb{R},\ \xi\in\mathbb{R}^N.\end{gathered} \tag{3.8}$$

Indeed (3.7) follows immediately from (3.5). To verify (3.8), one observes that by (3.5) one has, for each $t>0$,

$$A_i(x,ts,t\xi)\xi_i-t^{p-1}|\xi|^{p-2}\xi_i^2\ge -t^{p-1}c(t)|\xi_i|\left[|\xi|^{p-1}+|s|^{p-1}+K(x)\right], \tag{3.9}$$

and so

$$\sum_{i=1}^{N}A_i(x,ts,t\xi)\xi_i\ge t^{p-1}|\xi|^p\left[1-Nc(t)\left(1+\frac{2}{p}\right)\right]-\frac{1}{p'}t^{p-1}|c(t)|N\left(|s|^p+|K(x)|^{p'}\right). \tag{3.10}$$

Choosing t sufficiently large yields (3.8).

Remark 3.3. Equations (3.3) and (3.5) imply that A is well defined, continuous, and bounded from $W_0^{1,p}(\Omega)$ to $W^{-1,p'}(\Omega)$. Equations (3.3), (3.4), and (3.5) also imply that A is of type $(S)_+$. This latter fact can be proved along similar lines as in the argument given by Berkovits and Mustonen in [5].

We are now ready to state the following theorem.

Theorem 3.4. *Assume (2.2), (2.3), (2.5), (3.3), (3.4), and (3.5). Then for any $h \in W^{-1,p'}(\Omega)$, there exists a weak solution $u \in W_0^{1,p}(\Omega)$ of (3.1), that is,*

$$\int_\Omega \sum_{i=1}^N A_i(x,u(x),\nabla u(x))\frac{\partial v}{\partial x_i}\,dx = \int_\Omega f(x,u)v\,dx + \langle h,v\rangle, \quad \forall v \in W_0^{1,p}(\Omega). \tag{3.11}$$

Proof. The proof is rather similar to that of Theorem 2.3, and we will only detail below those points which really involve the operator A.

Let $(S_t)_{t\in[0,1]}$ be the family of operators from $W_0^{1,p}(\Omega)$ to $W^{-1,p'}(\Omega)$ defined by

$$S_t(u) = tA(u) - (1-t)(\Delta_p u) - t[f(x,u)+h(x)] - (1-t)\alpha|u|^{p-2}u, \tag{3.12}$$

for some fixed number α with $\lambda_1 < \alpha < \lambda_2$. Since the operator A is of type $(S)_+$, S_t is also of type $(S)_+$. By the degree theory for mappings of type $(S)_+$, as developed in Browder [7] and Berkowits and Mustonen [5], to solve (3.1) it suffices to prove the following estimate:

$$\exists R > 0 \text{ such that } \forall t \in [0,1], \quad \forall u \in \partial B(OR) \text{ such that } S_t(u) \neq 0. \tag{3.13}$$

To prove (3.13), we assume by contradiction that

$$\forall n \in \mathbb{N},\ \exists t_n \in [0,1], \exists u_n \in W_0^{1,p}(\Omega) \text{ with } \|u_n\|_{1,p} = n, \text{ such that } S_{t_n}(u_n) = 0. \tag{3.14}$$

Let $w_n = u_n/n$. We can extract from (w_n) a subsequence, still denoted by (w_n), which converges weakly in $W_0^{1,p}(\Omega)$, strongly in $L^p(\Omega)$ and a.e. in Ω to $w \in W_0^{1,p}(\Omega)$. We can also suppose that t_n converges to $t \in [0,1]$.

In the same manner as in the proof of Theorem 2.3, to obtain a contradiction, we use Lemmas 2.4, 2.6, and 2.7 (which do not involve the operator A) together with the following two lemmas. □

Lemma 3.5. $w \not\equiv 0$.

Proof. By (3.14) we have

$$\begin{aligned}\left\langle \frac{t_nA(u_n)}{n^{p-1}} - (1-t_n)\Delta_p w_n, w_n\right\rangle &= (1-t_n)\alpha\int_\Omega |w_n|^p\,dx \\ &\quad + t_n\left[\int_\Omega g_n(x)w_n(x)\,dx + \frac{1}{n^{p-1}}\langle h,w_n\rangle\right].\end{aligned} \tag{3.15}$$

Since

$$\begin{aligned}&\left|\left\langle \frac{t_nA(u_n)}{n^{p-1}}-t_n(-\Delta_p w_n),w_n\right\rangle\right|\\&\qquad\le n^{1-p}\int_\Omega\sum_{i=1}^N\left|A_i(x,u_n,n\nabla w_n)-n^{p-1}|\nabla w_n|^{p-2}\frac{\partial w_n}{\partial x_i}\right|\cdot\left|\frac{\partial w_n}{\partial x_i}\right|dx,\end{aligned}\tag{3.16}$$

using (3.5) and the fact that $\|w_n\|_{1,p}=1$, we obtain

$$\begin{aligned}&\left|\left\langle \frac{t_nA(u_n)}{n^{p-1}}-t_n(-\Delta_p w_n),w_n\right\rangle\right|\\&\qquad\le c(n)\left[||\nabla w_n||^{p/p'}_{L^p(\Omega)}+||w_n||^{p/p'}_{L^p(\Omega)}+\|K\|_{L^{p'}(\Omega)}\right]||w_n||_{1,p}\xrightarrow{n\to+\infty}0.\end{aligned}\tag{3.17}$$

Therefore

$$1=(1-t)\alpha\int_\Omega|w|^p\,dx+t\int_\Omega g(x)w(x)\,dx,\tag{3.18}$$

which clearly implies $w\not\equiv 0$. □

Lemma 3.6. *w is a solution of*

$$\begin{aligned}-\Delta_p w&=m|w|^{p-2}w\quad\text{in }\Omega,\\ w&=0\quad\text{on }\partial\Omega,\end{aligned}\tag{3.19}$$

where $m(x)=((1-t)\alpha+t\tilde{g}(x))$ and $\tilde{g}$ is defined in Lemma 2.7.

Proof. We first show that w is a solution of

$$\begin{aligned}-\Delta_p w&=(1-t)\alpha|w|^{p-2}w+tg\quad\text{in }\Omega,\\ w&=0\quad\text{on }\partial\Omega.\end{aligned}\tag{3.20}$$

Since $(-\Delta_p)(w_n)$ is bounded in $W^{-1,p'}(\Omega)$, there exists a subsequence, still denoted by (w_n), and a distribution $T\in W^{-1,p'}(\Omega)$, such that $(-\Delta_p)(w_n)$ converges weakly to T in $W^{-1,p'}(\Omega)$. In particular

$$\lim_{n\to+\infty}\langle-\Delta_p w_n,w\rangle=\langle T,w\rangle.\tag{3.21}$$

We also have

$$\begin{aligned}\langle-\Delta_p w_n,w_n-w\rangle&=(1-t_n)\alpha\int_\Omega|w_n|^{p-2}w_n(w_n-w)\,dx\\&\quad+t_n\left[\int_\Omega g_n(x)(w_n-w)\,dx+\frac{1}{n^{p-1}}\langle h,w_n-w\rangle\right]\\&\quad-\left\langle t_n\left[\frac{A(u_n)}{n^{p-1}}+\Delta_p w_n\right],w_n-w\right\rangle,\end{aligned}\tag{3.22}$$

and since, by (3.5),

$$\left|\left\langle t_n\left[\frac{A(u_n)}{n^{p-1}}+\Delta_p w_n\right], w_n-w\right\rangle\right| \\ \leq c(n)\left[\|\nabla w_n\|_{L^p(\Omega)}^{p/p'}+\|w_n\|_{L^p(\Omega)}^{p/p'}+\|K\|_{L^{p'}(\Omega)}\right]\|w_n-w\|_{1,p} \xrightarrow{n\to+\infty} 0, \tag{3.23}$$

we deduce

$$\lim_{n\to+\infty}\langle -\Delta_p w_n, w_n-w\rangle = 0. \tag{3.24}$$

The rest of the proof of Lemma 3.6 uses the fact that $(-\Delta_p)$ is of type (M) and is similar to the proof of Lemma 2.8. □

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