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ON AZUMAYA ALGEBRAS WITH A FINITE AUTOMORPHISM GROUP

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ABSTRACT. Let B be a ring with 1, C the center of B, and G a finite automorphism group of B. It is shown that if B is an Azumaya algebra such that $B = \oplus\sum_{g\in G} J_g$ where $J_g = \{b \in B \mid bx = g(x)b$ for all $x \in B\}$, then there exist orthogonal central idempotents $\{f_i \in C \mid i = 1,2,\dots,m$ for some integer $m\}$ and subgroups H_i of G such that $B = (\oplus\sum_{i=1}^{m} Bf_i) \oplus D$ where Bf_i is a central Galois algebra with Galois group $H_i|_{Bf_i} \cong H_i$ for each $i = 1,2,\dots,m$ and D is contained in C.

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1. Introduction. Let A be an Azumaya algebra, G a finite algebra automorphism group of A, and $J_g = \{a \in A \mid ax = g(x)a$ for all $x \in A\}$ for each $g \in G$. In [6], it was shown that $J_gJ_h = J_{gh}$ for all $g,h \in G$. In [2], let B be a separable algebra over a commutative ring R and G a finite algebra automorphism group of B. Assume that $B = \oplus\sum_{g\in G} J_g$ where J_g are similarly defined as for A. Then, B is a central Galois algebra with Galois group G if and only if for each $g \in G$, $J_gJ_{g^{-1}} = C$, the center of B. Thus, any Azumaya algebra B with a finite algebra automorphism group G such that $B = \oplus\sum_{g\in G} J_g$ is a central Galois algebra with Galois group G. By changing the algebra automorphism group G to a ring automorphism group G, the purpose of the present paper is to generalize the above fact. We will show that if B is an Azumaya C-algebra with a finite ring automorphism group G such that $B = \oplus\sum_{g\in G} J_g$, then there exist orthogonal central idempotents $\{f_i \in C \mid i = 1,2,\dots,m$ for some integer $m\}$ and subgroups H_i of G such that $B = (\oplus\sum_{i=1}^{m} Bf_i) \oplus Bf$ where Bf_i is a central Galois algebra with Galois group $H_i|_{Bf_i} \cong H_i$ for each $i = 1,2,\dots,m$, $f = 1 - \sum_{i=1}^{m} f_i$, and $Bf = Cf$. Since a Galois algebra B with Galois group G is an Azumaya algebra such that $B = \oplus\sum_{g\in G} J_g$, our result can be applied to Galois algebras. Moreover, if B is a separable extension of B^G such that $B = \oplus\sum_{g\in G} J_g$, then the direct summand Bf is a commutative Galois algebra with Galois group $G|_{Bf} \cong G$. An example is given to demonstrate the results and to illustrate that an Azumaya algebra B such that $B = \oplus\sum_{g\in G} J_g$ is not necessarily a Galois algebra with Galois group G.

2. Definitions and notations. Throughout, B will represent a ring with 1, C the center of B, G a ring automorphism group of B of order n for some integer n, and B^G the set of elements in B fixed under each element in G. We denote $J_g = \{b \in B \mid bx = g(x)b$ for all $x \in B\}$ and $I_g = BJ_g \cap C$ for each $g \in G$.

Let A be a subring of a ring B with the same identity 1. We denote $V_B(A)$ the commutator subring of A in B. We follow the definitions of a Galois extension, a separable extension, and an Azumaya algebra as given in [1, 5, 7]. The ring B is called a separable extension of A if there exist $\{a_i, b_i$ in B, $i = 1, 2, \ldots, m$ for some integer $m\}$ such that $\sum a_i b_i = 1$, and $\sum b a_i \otimes b_i = \sum a_i \otimes b_i b$ for all b in B where $\otimes$ is over A. An Azumaya algebra is a separable extension of its center. The ring B is called a Galois extension of B^G with Galois group G if there exist elements $\{a_i, b_i$ in B, $i = 1, 2, \ldots, m\}$ for some integer m such that $\sum_{i=1}^{m} a_i g(b_i) = \delta_{1,g}$ for each $g \in G$. The algebra B is called a Galois algebra over R if B is a Galois extension of R which is contained in C, and B is called a central Galois extension if B is a Galois extension of C.

3. The structure theorem. In this section, we assume that B is an Azumaya C-algebra with a finite ring automorphism group G such that $B = \oplus \sum_{g \in G} J_g$. We will show a structure theorem for such a B. We begin with some properties of the C-module J_g for $g \in G$ similar to those as for a Galois algebra (see [4, Proposition 2]).

Lemma 3.1. *For all $g, h \in G$,*

(1) $J_g J_h = I_g J_{gh} = I_h J_{gh}$ where $I_g = BJ_g \cap C$ and $I_h = BJ_h \cap C$.

(2) There is a unique idempotent $e_g \in C$ such that $BJ_g = Be_g$ and $J_g J_{g^{-1}} = e_g C$.

Proof. (1) Since B is an Azumaya C-algebra and BJ_g is an ideal of B, $BJ_g = BI_g$ (see [1, Proposition 1.11, page 46]). By hypothesis, $B = \oplus \sum_{g \in G} J_g$, so $BJ_h = \sum_{g \in G} J_g J_h$. Noting that $J_g J_h \subset J_{gh}$ and $B = \oplus \sum_{g \in G} J_{gh}$, we have that $BJ_h = \oplus \sum_{g \in G} J_g J_h$. Hence $\oplus \sum_{g \in G} J_g J_h = BJ_h = BI_h = \oplus \sum_{g \in G} J_{gh} I_h$. Thus, $J_g J_h = I_h J_{gh}$. Similarly, $J_g J_h = I_g J_{gh}$.

(2) By (1), $J_g J_h = I_g J_{gh}$ for all $g, h \in G$. By letting $h = 1$, we have $I_g J_g = J_g J_1 = J_g C = J_g$, and by letting $h = g^{-1}$, we have $J_g J_{g^{-1}} = I_g J_1 = I_g C = I_g$. Thus, $(I_g)^2 = I_g J_g J_{g^{-1}} = J_g J_{g^{-1}} = I_g$. Moreover, since $B = \oplus \sum_{g \in G} J_g$ is an Azumaya C-algebra, J_g is a finitely generated and projective C-module for each $g \in G$. Hence $BJ_g \cong B \otimes_C J_g$ is a finitely generated and projective ideal of B. This implies that $I_g (= BJ_g \cap C)$ is a finitely generated and projective ideal of C. But $(I_g)^2 = I_g$, so $I_g = Ce_g$ for some idempotent $e_g \in C$ (see [4, Lemma 2] and [3, Theorem 76]). Therefore, $BJ_g = BI_g = Be_g$ and $J_g J_{g^{-1}} = I_g = e_g C$. Since e_g is the identity of Be_g, it is unique. □

By Lemma 3.1(2), for each $g \in G$, there is a unique idempotent $e_g \in C$ such that $BJ_g = Be_g$. The Boolean algebra generated by the elements $\{e_g \mid g \in G$ and $BJ_g = Be_g\}$ is denoted by E.

Lemma 3.2. *Let e be a nonzero element in E of the form $e = \Pi_{h \in H} e_h$ for some maximum subset H of G. Then H is a subgroup of G and $h(e) = e$ for each $h \in H$.*

Proof. For any $g, h \in H$,

$$Be_g e_h = (BJ_g)(BJ_h) = B(J_g J_h) = B(I_g J_{gh}) = (BI_g)(BJ_{gh}) = Be_g e_{gh}. \tag{3.1}$$

Hence $e_g e_h = e_g e_{gh}$. Thus, $e_g e_h = e_g e_h^2 = e_g e_{gh} e_h$. Therefore, $e = ee_{gh}$. Thus, $gh \in H$ by the maximality of H. Since G is finite, that $gh \in H$ whenever $g, h \in H$ implies that H is a subgroup of G. Noting that, for a subgroup H, $gHg^{-1} = H$ for all $g \in H$, we have that

$$g(Be) = g(B(\Pi_{h \in H} J_h)) = B(\Pi_{h \in H} g(J_h)) = B(\Pi_{h \in H} J_{ghg^{-1}}) = B(\Pi_{h \in H} J_h) = Be \tag{3.2}$$

for each $g \in H$. Hence, $g(e) = e$ for each $g \in H$ because e is the identity of Be. □

Next we show that $H|_{Be}$ is an algebra automorphism group.

Lemma 3.3. *Let e be a nonzero element in E of the form $e = \Pi_{h\in H} e_h$ for some maximum subset H of G. Then h restricted to Ce is an identity for each $h \in H$.*

Proof. For any $h \in H$ and $b \in J_h$, $bc = h(c)b$ for all $c \in C$, so $(c-h(c))b = 0$. Hence $(c-h(c))J_h = \{0\}$. Therefore $B(c-h(c))e_h = (c-h(c))Be_h = (c-h(c))BJ_h = B(c-h(c))J_h = \{0\}$. Thus, $(c-h(c))e_h = 0$. But $e = \Pi_{h\in H} e_h$, so $(c-h(c))e = 0$. Moreover, $h(e) = e$ for each $h \in H$ by Lemma 3.2, so $0 = (c-h(c))e = (c-h(c))h(e) = ch(e) - h(c)h(e) = ce - h(ce)$, that is, $h(ce) = ce$ for all $c \in C$. □

Lemma 3.4. *Let $J_h^{(Bf)} = \{b \in Bf \mid bx = h(x)b$ for all $x \in Bf\}$ for any $f \in E$ and $h \in G$. If $h(f) = f$, then $J_h^{(Bf)} = fJ_h$.*

Proof. It is clear that $fJ_h \subset J_h^{(Bf)}$. Conversely, for any $b \in J_h^{(Bf)}$, $b = fb$ and $bx = h(x)b$ for each $x \in Bf$. Hence for any $y \in B$, $by = (fb)y = b(yf) = h(yf)b = h(y)fb = h(y)b$. Therefore, $b \in J_h$, and so $b = fb \in fJ_h$. Thus, $J_h^{(Bf)} = fJ_h$. □

Let e and H be given as in Lemma 3.2. We have a structure theorem for the Azumaya Ce-algebra Be with an algebra automorphism group $H|_{Be} \cong H$ and for the Azumaya C-algebra B with a ring automorphism group G, respectively.

Theorem 3.5. *Let e be a nonzero element in E of the form $e = \Pi_{h\in H} e_h$ for some maximum subset H of G. Then Be is a central Galois algebra with Galois group $H|_{Be} \cong H$.*

Proof. By Lemma 3.2, H is a subgroup of G and $h(e) = e$ for any $h \in H$. By Lemma 3.3, h restricted to Ce is an identity for each $h \in H$. Hence $H|_{Be}$ is a Ce-algebra automorphism group of Be. Since B is an Azumaya C-algebra, Be is an Azumaya Ce-algebra (see [1, Proposition 1.11, page 46]). By Lemma 3.4, $J_h^{(Be)} = eJ_h$ for each $h \in H$, so $Be = \oplus\sum_{g\in G} J_g e = \oplus\sum_{g\in H} eJ_g \oplus \sum_{g\notin H} eJ_g$. Since H is a maximum subset of G such that $e = \Pi_{h\in H} e_h$, $ee_g = 0$ for each $g \notin H$. This implies that $BeJ_g = Bee_g = \{0\}$. Therefore, $eJ_g = \{0\}$ for each $g \notin H$. Thus, $Be = \oplus\sum_{g\in H} eJ_g = \oplus\sum_{g\in H} J_g^{(Be)}$. Moreover, $J_h^{(Be)} J_{h^{-1}}^{(Be)} = (eJ_h)(eJ_{h^{-1}}) = eJ_hJ_{h^{-1}} = ee_hC = Ce$ which is the center of Be by Lemma 3.1. Thus, Be is a central Galois algebra over Ce with Galois group $H|_{Be}$ (see [2, Theorem 1]). Next, we claim that $H|_{Be} \cong H$. Since $e \neq 0$, $\{0\} \neq Be = Bee_h = BeJ_h = BJ_h^{(Be)}$ for each $h \in H$. Hence $J_h^{(Be)} \neq \{0\}$ for each $h \in H$. Now, if $h|_{Be} = 1$, then $\{0\} \neq Ce = J_h^{(Be)} = eJ_h \subset C \cap J_h = J_1 \cap J_h$. But $B = \oplus\sum_{g\in G} J_g$, so $J_1 = J_h$. Therefore $h = 1$. This implies that $h|_{Be} \neq 1$ whenever $h \neq 1$ in H. Thus, $H|_{Be} \cong H$. □

Theorem 3.6. *Let B be an Azumaya C-algebra with a finite ring automorphism group G such that $B = \oplus\sum_{g\in G} J_g$, then there exist orthogonal idempotents $\{f_i \in C \mid i = 1,2,\dots,m$ for some integer $m\}$ and subgroups H_i of G such that $B = (\oplus\sum_{i=1}^m Bf_i) \oplus Cf$ where Bf_i is a central Galois algebra with Galois group $H_i|_{Bf_i} \cong H_i$ for each $i = 1,2,\dots,m$ and $f = 1-\sum_{i=1}^m f_i$.*

Proof. Let $\{f_i \in E \mid i = 1,2,\dots,k\}$ be the set of all distinct nonzero elements in E of the form $f_i = \Pi_{h\in H_i} e_h$ for some maximum subset (subgroup) H_i of G as given in Lemma 3.2. Then they are orthogonal. Hence $B = (\oplus\sum_{i=1}^k Bf_i) \oplus Bf$ where $f = 1 - \sum_{i=1}^k f_i$ such that Bf_i is a central Galois algebra with Galois group $H_i|_{Bf_i} \cong H_i$ for each

$i = 1, 2, \ldots, k$ by Theorem 3.5. Next, we claim that $Bf = Cf$. Since $\{f_i \mid i = 1, 2, \ldots, k\}$ is the set of all distinct nonzero elements in E of the form $f_i = \Pi_{h \in H_i} e_h$ for some maximum subset (subgroup) H_i of G, g permutes the set $\{f_i \mid i = 1, 2, \ldots, k\}$ for each $g \in G$. Hence $g(f) = f$ for each $g \in G$. Hence, by Lemma 3.4, $J_g^{(Bf)} = fJ_g$ for each $g \in G$. Therefore, $Bf = \oplus \sum_{g \in G} J_g f = \oplus \sum_{g \in G} J_g^{(Bf)}$ is an Azumaya Cf-algebra with a finite ring automorphism group $G|_{Bf}$. If $J_g^{(Bf)} = \{0\}$ for each $g \neq 1$ in G, then $Bf = J_1^{(Bf)} = fJ_1 = Cf$, and so we are done. If $J_g^{(Bf)} \neq \{0\}$ for some $g \neq 1$ in G, we can repeat the above argument to have more direct summands of central Galois algebras. Since E is finite, we have only finitely many central orthogonal idempotents $\{f_i \in E \mid i = 1, 2, \ldots, m$ for some integer $m\}$ such that $B = (\oplus \sum_{i=1}^m Bf_i) \oplus Bf$ where Bf_i is a central Galois algebra with Galois group $H_i|_{Bf_i} \cong H_i$ for each $i = 1, 2, \ldots, m$ and $Bf = Cf$. This completes the proof. □

Remark 3.7. Theorem 3.6 generalizes the following theorem of Harada (see [2, Theorem 1]):

Let B be a separable R-algebra with automorphism group G. If $B = \oplus \sum_{g \in G} J_g$ and $J_g J_{g^{-1}} = C$ for each $g \in G$, then B is a central Galois algebra with Galois group G.

Remark 3.8. Any Galois algebra with Galois group G satisfies the conditions as given in Theorem 3.6. There are Azumaya C-algebras B such that $B = \oplus \sum_{g \in G} J_g$, but B is not a Galois algebra with Galois group G (see Example 3.11). However, for a Galois extension B of B^G with Galois group G, the condition that B is a Galois algebra with Galois group G and that $B = \oplus \sum_{g \in G} J_g$ are equivalent as given by the following proposition.

Proposition 3.9. *For a Galois extension B of B^G with Galois group G, B is a Galois algebra with Galois group G if and only if $B = \oplus \sum_{g \in G} J_g$.*

Proof. Since B is a Galois extension of B^G with Galois group G, $V_B(B^G) = \oplus \sum_{g \in G} J_g$ (see [4, Proposition 1]). Hence $B = \oplus \sum_{g \in G} J_g$ if and only if $V_B(B^G) = B$, that is, $B^G \subset C$. □

As an application of Theorem 3.6, we obtain a structure theorem for a separable extension B of B^G such that $B = \oplus \sum_{g \in G} J_g$.

Theorem 3.10. *Let B be a separable extension of B^G such that $B = \oplus \sum_{g \in G} J_g$, then there exist orthogonal idempotents $\{f_i \in C \mid i = 1, 2, \ldots, m$ for some integer $m\}$ and subgroups H_i of G such that $B = (\oplus \sum_{i=1}^m Bf_i) \oplus Bf$ where Bf_i is a central Galois algebra with Galois group $H_i|_{Bf_i} \cong H_i$ for each $i = 1, 2, \ldots, m$, $f = 1 - \sum_{i=1}^m f_i$, and $Bf = Cf$ is a commutative Galois algebra with Galois group $G|_{Bf} \cong G$ if $f \neq 0$.*

Proof. For any $a \in B^G$ and $b = \sum_{g \in G} b_g \in B$ where $b_g \in J_g$, $b_g a = g(a) b_g = ab_g$ for each $g \in G$, so $ba = \sum_{g \in G} b_g a = a \sum_{g \in G} b_g = ab$ for any $b \in B$. Thus, $a \in C$ for any $a \in B^G$. Therefore, $B^G \subset C$. Noting that B is a separable algebra over B^G, we have that B is an Azumaya C-algebra. But $B = \oplus \sum_{g \in G} J_g$, so, by Theorem 3.6, there exist orthogonal idempotents $\{f_i \in C \mid i = 1, 2, \ldots, m$ for some integer $m\}$ and subgroups H_i of G such that $B = (\oplus \sum_{i=1}^m Bf_i) \oplus Bf$ where Bf_i is a central Galois algebra with Galois group $H_i|_{Bf_i} \cong H_i$ for each $i = 1, 2, \ldots, m$, $Bf = Cf$, and $f = 1 - \sum_{i=1}^m f_i$. Thus, it suffices to show that $Bf (= Cf)$ is a commutative Galois algebra with Galois group

$G|_{Bf} \cong G$ in case $f \neq 0$. In fact, since B^G is contained in C and B is separable over B^G, C is separable over B^G (see [1, Theorem 3.8, page 55]), and so Cf is separable over $B^G f$ (see [1, Proposition 1.11, page 46]). Moreover, since $f \in C^G$, $B^G f \subset (B^G f)^G \subset (Cf)^G$. Hence Cf is separable over $(Cf)^G$ (see [1, Proposition 1.11, page 46]). Furthermore, by Lemma 3.4, $J_g^{(Cf)} = J_g^{(Bf)} = fJ_g$ for each $g \in G$ so $J_g^{(Cf)} \subset C \cap J_g = \{0\}$ for each $g \neq 1$ in G. This implies that $g|_{Cf} \neq$ identity whenever $g \neq 1$ in G (for $J_1^{(Cf)} = Cf$). Thus, $G|_{Bf} \cong G$ and $Bf(=Cf)$ is a commutative Galois algebra with Galois group $G|_{Bf} \cong G$ (see [2, Proposition 2]). This completes the proof. □

We conclude the present paper with an example to demonstrate the results in Theorem 3.6 and illustrate that an Azumaya C-algebra B such that $B = \oplus\sum_{g\in G} J_g$, but not necessarily a Galois algebra with Galois group G.

Example 3.11. Let $\mathbb{R}[i,j,k]$ be the real quaternion algebra over the field of real numbers $\mathbb{R}$, $\mathbb{Z}$ the integer ring, $D = (\mathbb{Z}+\sqrt{-1}\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}+\sqrt{-1}\mathbb{Z})$, $B = \mathbb{R}[i,j,k] \oplus D$, and $G = \{1, g_i, g_j, g_k\}$ where $g_i(a, d_1 \otimes d_2) = (iai^{-1}, \bar{d}_1 \otimes d_2)$, $g_j(a, d_1 \otimes d_2) = (jaj^{-1}, d_1 \otimes \bar{d}_2)$, and $g_k(a, d_1 \otimes d_2) = (kak^{-1}, \bar{d}_1 \otimes \bar{d}_2)$, for all $(a, d_1 \otimes d_2)$ in B, where $\bar{d}$ is the conjugate of the complex number d. Then,

(1) The center of B is $C = \mathbb{R} \oplus D$.

(2) B is an Azumaya C-algebra.

(3) $J_1 = C = \mathbb{R} \oplus D$, $J_{g_i} = \mathbb{R}(i,0)$, $J_{g_j} = \mathbb{R}(j,0)$, $J_{g_k} = \mathbb{R}(k,0)$. Hence $B = \oplus\sum_{g\in G} J_g$.

(4) By (3), $J_g J_{g^{-1}} = C(1,0)$ for each $g \neq 1$ in G. Hence, $f_1 = (1,0)$ is the only nonzero element in E of the form $f_1 = \Pi_{h\in H_1} e_h$ for some maximum subset H_1 of G (here $H_1 = G$) and $f = 1 - f_1 = (0, 1\otimes 1)$.

(5) $B = (\oplus\sum_{i=1}^{m} Bf_i) \oplus Cf$ where $m = 1$, Bf_i is a central Galois algebra with Galois group $H_i|_{Bf_i} \cong H_i$ for each $i = 1, 2, \ldots, m$.

(6) $B^G = \mathbb{R} \oplus (\mathbb{Z} \otimes \mathbb{Z}) = \mathbb{R} \oplus \mathbb{Z}$.

(7) Since D is not separable over $\mathbb{Z}$, B is not separable over $B^G(= \mathbb{R} \oplus \mathbb{Z})$. Hence B is not a Galois algebra with Galois group G.

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References

[1] F. DeMeyer and E. Ingraham, *Separable Algebras over Commutative Rings*, Lecture Notes in Mathematics, vol. 181, Springer-Verlag, Berlin, 1971. MR 43#6199. Zbl 215.36602.

[2] M. Harada, *Supplementary results on Galois extension*, Osaka J. Math. **2** (1965), 343–350. MR 33#151. Zbl 178.36903.

[3] T. Kanzaki, *On Galois algebra over a commutative ring*, Osaka J. Math. **2** (1965), 309–317. MR 33#150. Zbl 163.28802.

[4] I. Kaplansky, *Commutative Rings*, Allyn and Bacon, Massachusetts, 1970. MR 40#7234. Zbl 203.34601.

[5] P. Nuss, *Extensions galoisiennes non commutatives: normalité, cohomologie non abélienne* [*Noncommutative Galois extensions: normality, nonabelian cohomology*], Comm. Algebra **28** (2000), no. 7, 3223–3251 (French). MR 2001d:16050. Zbl 01475110.

[6] A. Rosenberg and D. Zelinsky, *Automorphisms of separable algebras*, Pacific J. Math. **11** (1961), 1109–1117. MR 26#6215. Zbl 116.02501.

[7] G. Szeto and L. Xue, *On characterizations of a center Galois extension*, Int. J. Math. Math. Sci. **23** (2000), no. 11, 753–758. MR 2001c:16061. Zbl 01501976.

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