IJMMS 26:2 (2001) 117–121
PII. S0161171201005397
http://ijmms.hindawi.com

NEAR FRATTINI SUBGROUPS OF RESIDUALLY FINITE GENERALIZED FREE PRODUCTS OF GROUPS

MOHAMMAD K. AZARIAN

(Received 3 May 1999 and in revised form 9 June 2000)

ABSTRACT. Let $G = A \star_H B$ be the generalized free product of the groups A and B with the amalgamated subgroup H. Also, let $\lambda(G)$ and $\psi(G)$ represent the lower near Frattini subgroup and the near Frattini subgroup of G, respectively. If G is finitely generated and residually finite, then we show that $\psi(G) \le H$, provided H satisfies a nontrivial identical relation. Also, we prove that if G is residually finite, then $\lambda(G) \le H$, provided: (i) H satisfies a nontrivial identical relation and A, B possess proper subgroups A_1, B_1 of finite index containing H; (ii) neither A nor B lies in the variety generated by H; (iii) $H < A_1 \le A$ and $H < B_1 \le B$, where A_1 and B_1 each satisfies a nontrivial identical relation; (iv) H is nilpotent.

2000 Mathematics Subject Classification. 20E06, 20E28.

1. Definitions and notation. Throughout the paper our notation will be standard. We use $G = A \star_H B$ to represent the generalized free product of the groups A and B with the amalgamated subgroup H, as in B. H. Neumann [12]. A group G is residually finite if every nontrivial element of G can be excluded from some normal subgroup of finite index in G. An N-group is a finite group in which the normalizer of every nontrivial solvable subgroup is solvable. A group G is called 3-metabelian if every subgroup of G which is generated by 3 elements is metabelian. A variety of groups is the collection of all groups satisfying a given set of identical relations or laws. The core of the subgroup H in G is represented by $K(G,H)$.

An element g of a group G is a near generator of G if there exists a subset S of G such that $|G : \langle S\rangle|$ is infinite, but $|G : \langle g, S\rangle|$ is finite. Thus, an element g of G is a non-near generator of G if for every subset S of G, finiteness of $|G : \langle g, S\rangle|$ implies finiteness of $|G : \langle S\rangle|$. A subgroup M of a group G is nearly maximal in G if $|G : M|$ is infinite, but $|G : N|$ is finite, whenever $M < N \le G$. That is, M is nearly maximal in G if it is maximal with respect to being of infinite index in G. The set of all non-near generators of a group G forms a characteristic subgroup called the lower near Frattini subgroup of G, denoted by $\lambda(G)$. The intersection of all nearly maximal subgroups of G is called the upper near Frattini subgroup of G, denoted by $\mu(G)$. If there are no nearly maximal subgroups, then $\mu(G) = G$. In general, $\lambda(G) \le \mu(G)$. If $\lambda(G) = \mu(G)$, then their common value is called the near Frattini subgroup of G, denoted by $\psi(G)$. Definitions concerning the near Frattini subgroup are due to J. B. Riles [13].

2. Background and history. In response to a question raised by N. Itô concerning the existence of maximal subgroups in free products of groups, G. Higman and B. H. Neumann proved that the Frattini subgroup of a free product of (nontrivial) groups is the trivial group [11, Theorem 2, page 87]. That is, they showed that free products of groups do have maximal subgroups. They extended Itô's question and asked whether a generalized free product of groups necessarily has maximal subgroups. They asked whether or not the Frattini subgroup of a generalized free product of groups is contained in the amalgamated subgroup. These questions have been answered for some certain classes of generalized free products of groups (see [2, 3, 4, 5, 6, 7, 8, 9]).

Similar results for the (lower) near Frattini subgroups of such generalized free products of groups are produced in [2, 3, 4, 5, 6, 7, 8, 9]. In this paper which is motivated by R. B. J. T. Allenby and C. Y. Tang [1], we continue our investigation to produce more results concerning the relationship between the (lower) near Frattini subgroup and the amalgamated subgroup of these generalized free products. In particular, in Section 3 we show that if $G = A\star_H B$ is finitely generated and residually finite, then $\psi(G) \le H$, provided H satisfies a nontrivial identical relation. Also, when $G = A\star_H B$ is residually finite, we prove that $\lambda(G) \le H$, if any of the following conditions is satisfied: (i) H satisfies a nontrivial identical relation and A, B possess proper subgroups A_1, B_1 of finite index containing H; (ii) neither A nor B lies in the variety generated by H; (iii) $H < A_1 \le A$ and $H < B_1 \le B$, where A_1 and B_1 each satisfies a nontrivial identical relation; (iv) H is nilpotent.

3. Results. Before tackling the new results, we need to state some known results from previous works.

Theorem 3.1 [5, Theorem 3.6, page 502]**.** *Let $G = A\star_H B$. If H satisfies the minimum condition on subgroups, then $\lambda(G) \le K(G,H)$.*

Theorem 3.2 [7, Theorem 4.2, page 6]**.** *Let $G = A\star_H B$. If there exists a nontrivial normal subgroup N of G such that $N \cap H = 1$, then $\lambda(G) \le H$.*

Proposition 3.3 [7, Proposition 4.6, page 6]**.** *Let $G = A\star_H B$. If $\lambda(G)\cap A = \lambda(G)\cap B = \lambda(G)\cap H$, then $\lambda(G) \le H$.*

Theorem 3.4 [7, Theorem 4.7, page 6]**.** *Let $G = A\star_H B$. Suppose A_1 and B_1 are finite normal subgroups of A and B, respectively. If $A_1 \cap H = B_1 \cap H$, and at least one of A_1 or B_1 is not contained in H, then $\lambda(G) \le H$.*

Theorem 3.5 [9, Theorem 3.12, page 608]**.** *Let $G = A\star_H B$ be residually finite. If $|A:H| = |B:H| = 2$, then $\lambda(G) \le K(G,H)$.*

Theorem 3.6 [7, Theorem 4.11, page 7]**.** *Let $G = A\star_H B$. If A and B are countable groups, then $\lambda(G) \le H$.*

Theorem 3.7. *Let $G = A\star_H B$ be finitely generated and residually finite. If H satisfies a nontrivial identical relation, then $\psi(G) \le H$.*

Remark 3.8. We could refer to Theorem 3.6 and accept Theorem 3.7 without a proof. However, we present a direct proof, independent of the proof of Theorem 3.6.

Proof. Since G is finitely generated by J. B. Riles [13, Proposition 1, page 157], $\lambda(G) = \mu(G) = \psi(G)$. Therefore, it is enough to show that $\lambda(G) \le H$. If $\lambda(G)\cap A =$

$\lambda(G)\cap B = \lambda(G)\cap H$, then Proposition 3.3 is applicable. Otherwise, at least one of $\lambda(G)\cap A$ or $\lambda(G)\cap B$ properly contains $\lambda(G)\cap H$. If $|A:H| = |B:H| = 2$, then by Theorem 3.5, $\lambda(G) \le H$. Therefore, without loss of generality, we may assume that $|A:H| > 2$. Thus, there must exist an element $a \in A$ such that $a \in \lambda(G)$ but $a \notin H$. Also, we let $b \in B\backslash H$ and $a_1 \in A\backslash H \cup aH$. Now, since $a_1^{-1}a \notin H$, we conclude that $u = a_1^{-1}(ab^{-1}ab)a_1 \in \lambda(G)$, where in reduced form the initial and final letters of u are in $A\backslash H$.

Since $\lambda(G)$ is characteristic in G, the rest of the proof is very similar to the proof of Theorem 2 of R. B. J. T. Allenby and C. Y. Tang [1, page 302]. Thus, we use the same notation and set up as in [1] and we replace $\Phi(G)$ by $\lambda(G)$. In particular, we let $S = \langle u, b^{-1}ub\rangle$, $w(x_1,x_2,\dots,x_n)$, $w(y_1,y_2,\dots,y_n)$, N, U, V, $\bar{A}$, $\bar{B}$, $\bar{H}$, $\bar{G}$, and the natural map ψ, be as in the proof of Theorem 2 of [1]. To complete the proof we use the fact that G is residually finite, Theorem 3.4, as well as the fact that the natural homomorphism takes a non-near generator of G to a non-near generator of $\bar{G}$. □

Theorem 3.7 can be applied to various residually finite generalized free products of groups. For example, if $G = A\star_H B$ is residually finite and is finitely generated, then $\psi(G) \le H$, provided: (i) H is of finite exponent, H is periodic or H is an N-group; (ii) H is the ordinary free product of two cyclic groups of order 2; (iii) H is metabelian, or H is 3-metabelian; (iv) H is nilpotent.

Theorem 3.9. *Let $G = A\star_H B$ be residually finite. If H satisfies a nontrivial identical relation and if A,B possess proper subgroups A_1,B_1 of finite index containing H, then $\lambda(G) \le H$.*

Proof. The first part of the proof is similar to the proof of Theorem 3 of R. B. J. T. Allenby and C. Y. Tang [1, page 302], and we note that A_1, B_1 here correspond to K, L in [1]. Thus, if $U = N\cap A\cap\lambda(G)\lhd A$ and $V = N\cap B\cap\lambda(G)\lhd B$, are as in [1], where $\Phi(G)$ is replaced by $\lambda(G)$, then it is enough to show that $HU \ne A$ and $HV \ne B$. If this is not the case, then without loss of generality, we may assume that $HU = A$. Now, from the fact that $H \le A_1 < A$ and $|A:A_1| < \infty$, we deduce that $A_1(\lambda(G)\cap A) \ge HU = A$. This implies that

$$A = \langle A_1, \lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n\rangle, \tag{3.1}$$

where $\lambda_1,\lambda_2,\lambda_3,\dots,\lambda_n$ are nontrivial and distinct elements of $\lambda(G)$. Thus,

$$G = \langle A, B\rangle = \langle \lambda_1,\lambda_2,\lambda_3,\dots,\lambda_n, A_1, B\rangle. \tag{3.2}$$

Hence,

$$\left|G : \langle \lambda_1,\lambda_2,\lambda_3,\dots,\lambda_n, A_1, B\rangle\right| < \infty. \tag{3.3}$$

But, since $\lambda_1,\lambda_2,\lambda_3,\dots,\lambda_n$ are non-near generators of G, we must have $|G:\langle A_1,B\rangle| < \infty$.

However, $|G:\langle A_1,B\rangle| < \infty$, is not possible. For if we take $a \in A\backslash A_1$ and $b \in B\backslash H$, then

$$ab\langle A_1,B\rangle, (ab)^2\langle A_1,B\rangle, \dots, (ab)^n\langle A_1,B\rangle, \dots \tag{3.4}$$

are incongruent mod$\langle A_1,B\rangle$. That is, $\langle A_1,B\rangle$ has infinitely many distinct cosets in G. Therefore, the assumption that $HU = A$ is reached to a contradiction, and thus, the proof is complete. □

Theorem 3.10. *Let $G = A \star_H B$ be residually finite. If neither A nor B lies in the variety generated by H, then $\lambda(G) \le H$.*

Proof. First we note that H must satisfy a nontrivial identical relation. Otherwise, H generates the variety of all groups, and thus it must contain both A and B, contradicting the statement of the theorem. Also, since G is residually finite, it contains a collection of normal subgroups $N_\lambda (\lambda \in \Lambda)$ of finite index such that $\bigcap_{\lambda \in \Lambda} N_\lambda = 1$. If there exist $\mu, \nu \in \Lambda$ such that $H(A \cap N_\mu) \ne A$ and $H(B \cap N_\nu) \ne B$, then by Theorem 3.9, $\lambda(G) \le H$. On the other hand, if $H(A \cap N_\lambda) = A$ for all $\lambda \in \Lambda$ and $H(B \cap N_\lambda) = B$, for all $\lambda \in \Lambda$, then again, by the argument given by R. B. J. T. Allenby and C. Y. Tang in the proof of the Frattini version of this theorem [1, Theorem 3, page 303], we conclude that $\lambda(G) \le H$. If $\lambda(G) \not\le H$, then we must have either $H(A \cap N_\lambda) = A$ for all $\lambda \in \Lambda$ or $H(B \cap N_\lambda) = B$, for all $\lambda \in A$, but not both. Hence, either A or B must satisfy the same identical relation as the amalgamated subgroup H, which is impossible, by the statement of the theorem. Therefore, we must have $\lambda(G) \le H$, as desired. □

Theorem 3.11. *Let $G = A \star_H B$ be residually finite. If $H < A_1 \le A$ and $H < B_1 \le B$, where A_1 and B_1 each satisfies a nontrivial identical relation, then $\lambda(G) \le H$.*

Proof. Since H satisfies a nontrivial identical relation, if G is finitely generated, then Theorem 3.6 is applicable. Also, if both A_1 and B_1 are of finite indices in A and B, respectively, then again by Theorem 3.9, $\lambda(G) \le H$. Now, since both $\lambda(G)$ and $\Phi(G)$ are characteristic subgroups of G, the proof of the general case is very similar to the proof of the Frattini version of this theorem by R. B. J. T. Allenby and C. Y. Tang [1, Theorem 1, page 303], and is left to the reader. □

Theorem 3.12. *Let $G = A \star_H B$ be residually finite. If H is nilpotent, then $\lambda(G) \le H$.*

Proof. If G is finitely generated, then Theorem 3.7 is applicable. Otherwise, we use the same setup and notation as in the proof of the Frattini version of this theorem [1, Theorem 5, page 303] by R. B. J. T. Allenby and C. Y. Tang. To complete the proof, we use the fact that the natural homomorphism ψ takes a non-near generator of G to a non-near generator of its factor group $\bar{G}$, and we apply Theorem 3.1 as well. □

As an immediate consequence of Theorem 3.12 and Theorem 7 of G. Baumslag [10, page 196], we have the following corollary.

Corollary 3.13. *Let $G = A \star_H B$. If A and B are free groups and H is cyclic, then $\lambda(G) \le H$.*

From our study of residually finite generalized free products of groups and their lower near Frattini subgroups in this paper, as well as [8, 9], we suspect that if the amalgamated subgroup satisfies a nontrivial identical relation, then the lower near Frattini subgroup of such generalized free products is contained in the amalgamated subgroup. Therefore, we make the following conjecture.

Conjecture 3.14. *Let $G = A \star_H B$ be residually finite. If H satisfies a nontrivial identical relation, then $\lambda(G) \le H$.*

Acknowledgement. This work was supported in part by a grant from the University of Evansville Faculty Fellowship.

References

[1] R. B. J. T. Allenby and C. Y. Tang, *On the Frattini subgroup of a residually finite generalized free product*, Proc. Amer. Math. Soc. **47** (1975), 300–304. MR 52#10892. Zbl 322.20018.

[2] M. K. Azarian, *On the near Frattini subgroups of amalgamated free products of groups*, Houston J. Math. **16** (1990), no. 4, 523–528. MR 92d:20032. Zbl 742.20030.

[3] ______, *On the lower near Frattini subgroups of generalized free products with cyclic amalgamations*, Houston J. Math. **17** (1991), no. 3, 419–423. MR 92j:20018a. Zbl 743.20020.

[4] ______, *On the near Frattini subgroup of the amalgamated free product of finitely generated abelian groups*, Houston J. Math. **17** (1991), no. 3, 425–427. MR 92j:20018b. Zbl 743.20021.

[5] ______, *On the lower near Frattini subgroups of amalgamated free products of groups*, Houston J. Math. **19** (1993), no. 4, 499–504. MR 94k:20046. Zbl 799.20028.

[6] ______, *On the near Frattini subgroups of certain groups*, Houston J. Math. **20** (1994), no. 3, 555–560. MR 95d:20046. Zbl 817.20028.

[7] ______, *A key theorem on the near Frattini subgroups of generalized free product of groups*, Houston J. Math. **22** (1996), no. 1, 1–10. MR 98b:20036. Zbl 855.20026.

[8] ______, *On the near Frattini subgroup of the generalized free product of finitely generated nilpotent groups*, Houston J. Math. **23** (1997), no. 4, 613–615. MR 2000e:20052a. Zbl 896.20019.

[9] ______, *On the near Frattini subgroups of amalgamated free products with residual properties*, Houston J. Math. **23** (1997), no. 4, 603–612. MR 2000e:20052b. Zbl 896.20020.

[10] G. Baumslag, *On the residual finiteness of generalized free products of nilpotent groups*, Trans. Amer. Math. Soc. **106** (1963), 193–209. MR 26#2489. Zbl 112.25904.

[11] G. Higman and B. H. Neumann, *On two questions of Itô*, J. London Math. Soc. **29** (1954), 84–88. MR 15,286g. Zbl 055.01602.

[12] B. H. Neumann, *An essay on free products of groups with amalgamations*, Philos. Trans. Roy. Soc. London Ser. A. **246** (1954), 503–554. MR 16,10d. Zbl 057.01702.

[13] J. B. Riles, *The near Frattini subgroups of infinite groups*, J. Algebra **12** (1969), 155–171. MR 39#322. Zbl 182.03702.

Mohammad K. Azarian: Department of Mathematics, University of Evansville, 1800 Lincoln Avenue, Evansville, IN 47722, USA

E-mail address: azarian@evansville.edu