IJMMS 26:5 (2001) 317–319
PII. S0161171201004550
http://ijmms.hindawi.com

SUFFICIENT CONDITIONS FOR MEROMORPHIC STARLIKENESS AND CLOSE-TO-CONVEXITY OF ORDER α

NAK EUN CHO and SHIGEYOSHI OWA

(Received 7 February 2000)

Abstract. The object of the present paper is to derive a property of certain meromorphic functions in the punctured unit disk. Our main theorem contains certain sufficient conditions for starlikeness and close-to-convexity of order α of meromorphic functions.

2000 Mathematics Subject Classification. 30C45.

1. Introduction. Let $\sum$ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the punctured unit disk $\mathcal{D} = \{z : 0 < |z| < 1\}$. A function $f \in \sum$ is said to be meromorphic starlike of order α if it satisfies

$$-\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in \mathcal{U} = \mathcal{D} - \{0\}) \tag{1.2}$$

for some α $(0 \le \alpha < 1)$. We denote by $\sum^*(\alpha)$ the class of all meromorphic starlike functions of order α.

Let $\mathrm{MC}(\alpha)$ be the subclass of $\sum$ consisting of functions f which satisfy

$$-\operatorname{Re}\{z^2 f'(z)\} > \alpha \quad (z \in \mathcal{U}) \tag{1.3}$$

for some α $(0 \le \alpha < 1)$. A function f in $\mathrm{MC}(\alpha)$ is meromorphic close-to-convex of order α in $\mathcal{D}$ (see [1]).

2. Main result. In proving our main theorem, we need the following lemma due to Owa, Nunokawa, Saitoh, and Fukui [2].

Lemma 2.1. *Let p be analytic in $\mathcal{U}$ with $p(0) = 1$. Suppose that there exists a point $z_0 \in \mathcal{U}$ such that* $\operatorname{Re} p(z) > 0$ $(|z| < |z_0|)$, $\operatorname{Re} p(z_0) = 0$, *and* $p(z) \ne 0$. *Then we have* $p(z) = ia(a \ne 0)$ *and*

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\frac{k}{2}\left(a + \frac{1}{a}\right), \tag{2.1}$$

where k is a real number with $k \ge 1$.

With the aid of Lemma 2.1, we derive the following theorem.

Theorem 2.2. *If $f \in \sum$ satisfies $f(z)f'(z) \neq 0$ in $\mathfrak{D}$ and*

$$\operatorname{Re}\left\{\alpha\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)}\right\} < 2(2-\alpha)-\beta \quad (z \in \mathcal{U}), \tag{2.2}$$

then

$$-\operatorname{Re}\left\{\frac{z^{2-\alpha}f'(z)}{f^{\alpha}(z)}\right\} > \frac{1}{1+2(2-\alpha)-2\beta} \quad (z \in \mathcal{U}), \tag{2.3}$$

where $\alpha \leq 2$ and $(2(2-\alpha)-1)/2 \leq \beta < 2-\alpha$.

Proof. We define the function p in $\mathcal{U}$ by

$$-\frac{z^{2-\alpha}f'(z)}{f^{\alpha}(z)} = \gamma + (1-\gamma)p(z) \tag{2.4}$$

with $\gamma = 1/(1+2(2-\alpha)-2\beta)$. Then p is analytic in $\mathcal{U}$ with $p(0) = 1$ and

$$\alpha\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)} = 2-\alpha-\frac{(1-\gamma)zp'(z)}{\gamma+(1-\gamma)p(z)}. \tag{2.5}$$

Suppose that there exists a point $z_0 \in \mathcal{U}$ such that

$$\operatorname{Re}p(z) > 0 \quad (|z| < |z_0|), \quad \operatorname{Re}p(z_0) = 0, \quad p(z) \neq 0. \tag{2.6}$$

Then, applying Lemma 2.1, we have $p(z) = ia(a \neq 0)$ and

$$\frac{z_0p'(z_0)}{p(z_0)} = i\frac{k}{2}\left(a+\frac{1}{a}\right) \quad (k \geq 1). \tag{2.7}$$

It follows from this that

$$\alpha\frac{z_0f'(z_0)}{f(z_0)} - \frac{z_0f''(z_0)}{f'(z_0)} = 2-\alpha-\frac{(1-\gamma)z_0p'(z_0)}{\gamma+(1-\gamma)p(z_0)} = 2-\alpha+\frac{k(1-\gamma)(1+a^2)}{2(\gamma+i(1-\gamma)a)}. \tag{2.8}$$

Therefore, we have

$$\begin{aligned}\operatorname{Re}\left\{\alpha\frac{z_0f'(z_0)}{f(z_0)} - \frac{z_0f''(z_0)}{f'(z_0)}\right\} &= 2-\alpha+\frac{k(1-\gamma)(1+a^2)}{2(\gamma^2+(1-\gamma)^2a^2)} \\ &\geq 2-\alpha+\frac{k(1-\gamma)}{2\gamma} \geq 2(2-\alpha)-\beta.\end{aligned} \tag{2.9}$$

This contradicts our assumption. Thus, we conclude that $\operatorname{Re}p(z) > 0$ for all $z \in \mathcal{U}$, that is,

$$-\operatorname{Re}\left\{\frac{z^{2-\alpha}f'(z)}{f^{\alpha}(z)}\right\} > \gamma = \frac{1}{1+2(2-\alpha)-2\beta} \quad (z \in \mathcal{U}). \tag{2.10}$$

Putting $\beta = (2(2-\alpha)-1)/2$ in Theorem 2.2, we have

Corollary 2.3. *If $f \in \sum$ satisfies $f(z)f'(z) \neq 0$ in $\mathfrak{D}$ and*

$$\operatorname{Re}\left\{\alpha\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)}\right\} < \frac{3}{2}-\alpha \quad (z \in \mathcal{U}), \tag{2.11}$$

then

$$-\operatorname{Re}\left\{\frac{z^{2-\alpha}f'(z)}{f^{\alpha}(z)}\right\} > \frac{1}{2} \quad (z \in \mathcal{U}), \tag{2.12}$$

where $\alpha \leq 2$.

Taking $\alpha = 1$ in Theorem 2.2, we have the following corollary.

Corollary 2.4. *If* $f \in \sum$ *satisfies* $f(z)f'(z) \neq 0$ *in* $\mathfrak{D}$ *and*

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)}\right\} < 2-\beta \quad (z \in \mathcal{U}), \tag{2.13}$$

then

$$-\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{1}{3-2\beta} \quad (z \in \mathcal{U}), \tag{2.14}$$

that is, $f \in \sum^*(1/(3-2\beta))$, *where* $1/2 \leq \beta < 1$.

Further, letting $\alpha = 0$ in Theorem 2.2, we have the following corollary.

Corollary 2.5. *If* $f \in \sum$ *satisfies* $f(z)f'(z) \neq 0$ *in* $\mathfrak{D}$ *and*

$$-\mathrm{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} < 4-\beta \quad (z \in \mathcal{U}), \tag{2.15}$$

then

$$-\mathrm{Re}\{z^2 f'(z)\} > 5-2\beta \quad (z \in \mathcal{U}), \tag{2.16}$$

that is, $f \in \mathrm{MC}(1/(5-2\beta))$, *where* $3/2 \leq \beta < 2$. □

Acknowledgement. This work was supported by the Korea Research Foundation Grant (KRF-99-015-DP0019).

References

[1] M. D. Ganigi and B. A. Uralegaddi, *Subclasses of meromorphic close-to-convex functions*, Bull. Math. Soc. Sci. Math. R. S. Roumanie (N.S.) **33(81)** (1989), no. 2, 105–109. MR 91b:30036. Zbl 685.30004.

[2] S. Owa, M. Nunokawa, H. Saitoh, and S. Fukui, *Starlikeness and close-to-convexity of certain analytic functions*, Far East J. Math. Sci. **2** (1994), no. 2, 143–148. MR 97d:30014. Zbl 933.30008.

Nak Eun Cho: Department of Applied Mathematics, Pukyong National University, Pusan 605-737, Korea

E-mail address: necho@dolphin.pknu.ac.kr

Shigeyoshi Owa: Department of Mathematics, Kinki University, Higashi-Osaka, Osaka 577-8502, Japan

E-mail address: owa@math.kindai.ac.jp