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ITERATIVE SOLUTIONS OF K-POSITIVE DEFINITE OPERATOR EQUATIONS IN REAL UNIFORMLY SMOOTH BANACH SPACES

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ABSTRACT. Let X be a real uniformly smooth Banach space and let $T : D(T) \subseteq X \to X$ be a K-positive definite operator. Under suitable conditions we establish that the iterative method by Bai (1999) converges strongly to the unique solution of the equation $Tx = f$, $f \in X$. The results presented in this paper generalize the corresponding results of Bai (1999), Chidume and Aneke (1993), and Chidume and Osilike (1997).

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1. Introduction and preliminaries. Let X be a real Banach space with a dual space X^*. The normalized duality mapping $J : X \to 2^{X^*}$ is defined by

$$J(x) = \{f \in X^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \quad x \in X. \tag{1.1}$$

It is known that X is uniformly smooth (equivalently, X^* is uniformly convex) if and only if J is single-valued and uniformly continuous on any bounded subset of X.

Chidume and Aneke [3] introduced the concept of K-positive definite operators and established the existence of the unique solution of the equation $Tx = f$ for that operator in real separable Banach spaces. Meanwhile they constructed, in L_p (or l_p) spaces with $p \ge 2$, an iteration method which converges strongly to the unique solution, provided that T and K commute. Chidume and Osilike [5] gave a new iteration scheme, in separable q-uniformly smooth Banach spaces, which converges strongly to the unique solution of the equation $Tx = f$, $f \in X$.

Recently, Bai [1] constructed a more general iteration procedure and improved the results of [3, 5] to separable uniformly smooth real Banach spaces.

Very recently, Zhou et al. [7] established the following excellent result, which is a generalization of the main result of Chidume and Aneke [3].

Lemma 1.1 (see [7]). *Let X be a real Banach space and let T be a K-positive definite operator with $D(T) = D(K)$. Then there exists a constant $\alpha > 0$ such that*

$$\|Tx\| \le \alpha \|Kx\|, \quad x \in D(T). \tag{1.2}$$

Moreover, the operator T is closed, $R(T) = X$, and the equation $Tx = f$ for each $f \in X$, has a unique solution.

The purpose of this paper is to study the convergence problem of the iteration procedure introduced in [1] for K-positive definite operators in real uniformly smooth real

Banach spaces. Our results extend the corresponding results due to Bai [1], Chidume and Aneke [3], and Chidume and Osilike [5].

In what follows, we will also need the following concepts and results.

Definition 1.2 (see [3, 7]). Let X be a real Banach space and X_1 a subspace of X. An operator T with domain $D(T) \supseteq X_1$ is called *continuously X_1-invertible* if T, as an operator restricted to X_1, has a bounded inverse on $R(T)$. A linear unbounded operator T with domain $D(T)$ in X and range $R(T)$ in X is called *K-positive definite* if there exist a continuously $D(T)$-invertible closed linear operator K with $D(A) \subseteq D(K)$ and a constant $c > 0$ such that

$$\langle Tu, j(Ku)\rangle \geq c\|Ku\|^2, \quad u \in D(T),\ j(Ku) \in J(Ku). \tag{1.3}$$

Let X be a real Banach space. Recall that the modulus of smoothness of X is defined by

$$\rho_X(t) = \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : x, y \in X,\ \|x\| = 1,\ \|y\| \leq t\right\}, \quad t \geq 0. \tag{1.4}$$

X is said to be *uniformly smooth* if $\lim_{t\to 0}\rho_X(t)/t = 0$. Let $p > 1$ be a real number. X is called *p-uniformly smooth* if there exists a constant $r > 0$ such that

$$\rho_X(t) \leq rt^p, \quad t > 0. \tag{1.5}$$

Hilbert spaces, L_p (or l_p) spaces, $1 < p < \infty$, and the Sobolev spaces W_p^m, $1 < p < \infty$, are all p-uniformly smooth. It is well known that the class of p-uniformly smooth real Banach spaces is a proper subclass of that of uniformly smooth real ones.

Lemma 1.3 (see [4, 6]). *Let X be a real uniformly smooth Banach space. Then*

(i) *there exist some positive constants A and B such that*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x)\rangle + A\max\{\|x\| + \|y\|, B\}\rho_X(\|y\|), \quad x, y \in X. \tag{1.6}$$

(ii) *there exists a continuous nondecreasing function $b : [0,\infty) \to [0,\infty)$ such that*

$$\begin{gathered} b(0) = 0, \qquad b(ct) \leq cb(t), \quad c \geq 1; \\ \|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x)\rangle + \max\{\|x\|, 1\}\|y\|b(\|y\|), \quad x, y \in X. \end{gathered} \tag{1.7}$$

Lemma 1.4 (see [2]). *Suppose that $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, and $\{\omega_n\}_{n=0}^{\infty}$ are nonnegative sequences such that*

$$\alpha_{n+1} \leq (1-\omega_n)\alpha_n + \beta_n\omega_n, \quad n \geq 0, \tag{1.8}$$

with $\{\omega_n\}_{n=0}^{\infty} \subset [0,1]$, $\sum_{n=0}^{\infty}\omega_n = \infty$ and $\lim_{n\to\infty}\beta_n = 0$. Then $\lim_{n\to 0}\alpha_n = 0$.

Lemma 1.5 (see [6]). *Let X be a real Banach space. Then*

(i) *$\rho_X(0) = 0$, $\rho_X(t) \leq t$, $t > 0$;*

(ii) *$\rho_X(t)$ is convex, continuous, and nondecreasing on $[0,\infty)$;*

(iii) *$\rho_X(t)/t$ is nondecreasing on $(0,\infty)$.*

2. Main results

Theorem 2.1. *Let X be a real uniformly smooth Banach space and let $T : D(T) \subseteq X \to X$ be a K-positive definite operator with $D(T) = D(K)$. Define a sequence $\{x_n\}_{n=0}^{\infty}$ iteratively from any $f \in X$ and $x_0 \in D(T)$ by*

$$y_n = x_n + b_n v_n, \quad x_{n+1} = y_n + a_n u_n, \quad n \geq 0; \tag{2.1}$$

$$v_n = K^{-1}f - K^{-1}Tx_n, \quad u_n = K^{-1}f - K^{-1}Ty_n, \quad n \geq 0, \tag{2.2}$$

where $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are arbitrary nonnegative sequences such that

$$\sum_{n=0}^{\infty} (a_n + b_n) = \infty; \tag{2.3}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0; \tag{2.4}$$

$$\max\{a_n, b_n\} \leq \frac{1}{2c}, \quad n \geq 0; \tag{2.5}$$

$$\alpha A \max\{(1+\alpha a_n)\|Kv_0\|,\ (1+\alpha b_n)\|Kv_0\|, B\} \leq 2c\|Kv_0\|, \quad n \geq 0, \tag{2.6}$$

where c, α, A and B are the constants appearing in (1.2), (1.3), and (1.6), respectively. Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique solution of the equation $Tx = f$.

Proof. It follows from Lemma 1.1 that the equation $Tx = f$ has a unique solution in X. Note that T and K are linear. From (2.1) and (2.2) we have

$$Kv_{n+1} = f - Tx_{n+1} = Ku_n - a_n Tu_n, \quad n \geq 0; \tag{2.7}$$

$$Ku_n = f - Ty_n = Kv_n - b_n Tv_n, \quad n \geq 0. \tag{2.8}$$

In view of (2.8) and (1.2), (1.3), and (1.6), we conclude that

$$\begin{aligned} \|Ku_n\|^2 &= \|Kv_n - b_n Tv_n\|^2 \\ &\leq \|Kv_n\|^2 - 2b_n\langle Tv_n, J(Kv_n)\rangle \\ &\quad + A\max\{\|Kv_n\| + b_n\|Tv_n\|, B\}\rho_X(b_n\|Tv_n\|) \\ &\leq (1-2cb_n)\|Kv_n\|^2 \\ &\quad + A\max\{(1+\alpha b_n)\|Kv_n\|, B\}\rho_X(\alpha b_n\|Kv_n\|) \end{aligned} \tag{2.9}$$

for all $n \geq 0$. Using (2.7) and (1.2), (1.3), and (1.6), we have

$$\begin{aligned} \|Kv_{n+1}\|^2 &= \|Ku_n - a_n Tu_n\|^2 \\ &\leq \|Ku_n\|^2 - 2a_n\langle Tu_n, J(Ku_n)\rangle \\ &\quad + A\max\{\|Ku_n\| + a_n\|Tu_n\|, B\}\rho_X(a_n\|Tu_n\|) \\ &\leq (1-2ca_n)\|Ku_n\|^2 \\ &\quad + A\max\{(1+\alpha a_n)\|Ku_n\|, B\}\rho_X(\alpha a_n\|Ku_n\|) \end{aligned} \tag{2.10}$$

for all $n \ge 0$. Set $M = \|Kv_0\|$. We claim that

$$\max\{||Kv_n||, ||Ku_n||\} \le M, \quad n \ge 0. \tag{2.11}$$

By virtue of (2.6), (2.9), and Lemma 1.5, we get that

$$\begin{aligned} ||Ku_0||^2 &\le (1-2cb_0)||Kv_0||^2 \\ &\quad + A\max\{(1+\alpha b_0)||Kv_0||, B\}\rho_X(\alpha b_0||Kv_0||) \\ &\le (1-2cb_0)M^2 + A\max\{(1+\alpha b_0)M, B\}\alpha b_0 M \\ &\le M^2. \end{aligned} \tag{2.12}$$

That is, (2.11) is true for $n = 0$. Suppose that (2.11) holds for some $n \ge 0$. Using (2.10), (2.6), and Lemma 1.5, we infer that

$$\begin{aligned} ||Kv_{n+1}||^2 &\le (1-2ca_n)||Ku_n||^2 \\ &\quad + A\max\{(1+\alpha a_n)||Ku_n||, B\}\rho_X(\alpha a_n||Ku_n||) \\ &\le (1-2ca_n)M^2 + A\max\{(1+\alpha a_n)M, B\}\alpha a_n M \\ &\le M^2. \end{aligned} \tag{2.13}$$

From (2.6), (2.9), (2.13), and Lemma 1.5, we have

$$\begin{aligned} ||Ku_{n+1}||^2 &\le (1-2cb_{n+1})||Kv_{n+1}||^2 \\ &\quad + A\max\{(1+\alpha b_{n+1})||Kv_{n+1}||, B\}\rho_X(\alpha b_{n+1}||Kv_{n+1}||) \\ &\le (1-2cb_{n+1})M^2 + A\max\{(1+\alpha b_{n+1})M, B\}\alpha b_{n+1} M \\ &\le M^2. \end{aligned} \tag{2.14}$$

Therefore (2.11) holds for all $n \ge 0$. Since X is uniformly smooth, by (2.4) and Lemma 1.5 we conclude that there exist nonnegative sequences $\{s_n\}_{n=0}^{\infty}$ and $\{t_n\}_{n=0}^{\infty}$ such that $\rho_X(\alpha M a_n) = s_n a_n$, $\rho_X(\alpha M b_n) = t_n b_n$ for all $n \ge 0$ and

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = 0. \tag{2.15}$$

It follows from (2.5), (2.9), (2.10), and (2.11) that

$$\begin{aligned} ||Kv_{n+1}||^2 &\le (1-2ca_n)(1-2cb_n)||Kv_n||^2 \\ &\quad + (1-2ca_n)A\max\{(1+\alpha b_n)||Kv_n||, B\}\rho_X(\alpha b_n||Kv_n||) \\ &\quad + A\max\{(1+\alpha a_n)||Ku_n||, B\}\rho_X(\alpha a_n||Ku_n||) \\ &\le [1-2c(a_n+b_n)+4c^2a_nb_n]||Kv_n||^2 \\ &\quad + A\max\{(1+\alpha)M, B\}(\rho_X(\alpha M a_n)+\rho_X(\alpha M b_n)) \\ &\le [1-c(a_n+b_n)]||Kv_n||^2 + L(a_ns_n+b_nt_n) \end{aligned} \tag{2.16}$$

for all $n \ge 0$, where $L = A\max\{(1+\alpha)M, B\}$. Let

$$\alpha_n = ||Kv_n||^2, \quad \omega_n = c(a_n+b_n), \quad \beta_n = \frac{L}{c}r_n, \quad n \ge 0, \tag{2.17}$$

where

$$r_n = \begin{cases} 0, & a_n + b_n = 0, \\ \dfrac{a_n}{a_n+b_n} s_n + \dfrac{b_n}{a_n+b_n} t_n, & a_n + b_n \neq 0. \end{cases} \tag{2.18}$$

It follows from (2.15) that $\lim_{n\to\infty} r_n = 0$. That is, $\lim_{n\to\infty} \beta_n = 0$. Thus (2.15) can be rewritten in the form

$$\alpha_{n+1} \leq (1-\omega_n)\alpha_n + \omega_n \beta_n, \quad n \geq 0. \tag{2.19}$$

Note that (2.3) and (2.5) mean that $\sum_{n=0}^{\infty} \omega_n = \infty$, $\omega_n \in [0,1]$. Consequently, Lemma 1.4 ensures that $\alpha_n \to 0$ as $n \to \infty$. That is,

$$\|Kv_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.20}$$

It follows from (2.2) and (2.20) that

$$\|Tx_n - f\| = \|Kv_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.21}$$

Note that T has a bounded inverse. Thus (2.21) means that $x_n \to T^{-1}f$, the unique solution of $Tx = f$. This completes the proof. □

Theorem 2.2. *Let X, T, K, f, $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$ and $\{u_n\}_{n=0}^{\infty}$ be as in Theorem 2.1. Suppose that $\{a_n\}$ and $\{b_n\}_{n=0}^{\infty}$ are any nonnegative sequences such that (2.3), (2.4), and (2.5) and*

$$\max\{b(\alpha a_n), b(\alpha b_n)\} \leq \frac{2c}{\max\{1, \|Kv_0\|\}}, \quad n \geq 0, \tag{2.22}$$

where $b(t)$ is as in (1.7), α and c are the constants appearing in (1.3) and (1.2), respectively. Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique solution of the equation $Tx = f$.

Proof. Set $M = \max\{1, \|Kv_0\|\}$. As in the proof of Theorem 3 in [1] we have

$$\|Kv_{n+1}\|^2 \leq (1 - c(a_n + b_n))\|Kv_n\|^2 + M^3\alpha(a_n b(\alpha a_n) + b_n b(\alpha b_n)), \quad n \geq 0. \tag{2.23}$$

Let

$$\alpha_n = \|Kv_n\|^2, \quad \omega_n = c(a_n + b_n), \quad \beta_n = \frac{\alpha}{c} M^3 r_n, \quad n \geq 0, \tag{2.24}$$

where

$$r_n = \begin{cases} 0, & a_n + b_n = 0, \\ \dfrac{a_n}{a_n+b_n} b(\alpha a_n) + \dfrac{b_n}{a_n+b_n} b(\alpha b_n), & a_n + b_n \neq 0. \end{cases} \tag{2.25}$$

It is easily seen that $\lim_{n\to\infty} \beta_n = 0$. The rest of the argument now follows as in the proof of Theorem 2.1 to yield that $x_n \to T^{-1}f$ as $n \to \infty$. This completes the proof. □

Remark 2.3. Theorems 2.1 and 2.2 extend Theorem 3.3 of Bai [1], Theorem 2 of Chidume and Aneke [3] and Theorem of Chidume and Osilike [5], respectively, in the following ways:

(a) Condition (2.3) is much weaker than $\sum_{n=0}^{\infty} a_n = \infty$ of [1].

(b) L_p (or l_p) spaces, $p \geq 2$, in [3] and q-uniformly smooth Banach space, $q > 1$, in [5] are replaced by the more general uniformly smooth Banach spaces.

(c) The commutativity condition of T and K in [3] is dropped.

(d) The iteration methods in [3, 5] are special cases of our iteration method.

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