POSITIVE SOLUTIONS FOR NONLINEAR
ELASTIC BEAM MODELS

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ABSTRACT. We give a negative answer to a conjecture of Korman on nonlinear elastic beam models. Moreover, by modifying the main conditions in the conjecture (generalizing the original ones at some points), we get positive results, that is, we obtain the existence of positive solutions for the models.

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1. Introduction. In 1988, Korman [3] studied the nonlinear elastic beam models

$$u'''' = f(x, u), \quad 0 < x < 1,$$
$$u(0) = \alpha, \quad u'(0) = \beta, \quad u(1) = \gamma, \quad -u'(1) = \delta,$$
(1.1)

where $\alpha, \beta, \gamma, \delta \geq 0$, $f : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous. Equation $u'''' = f(x, u)$ with appropriate boundary conditions describes various physically important boundary value problems (see [3]). Korman [3] studied the existence of positive solutions of (1.1) by using monotone iterations. His results covered in particular the sublinear nonlinearities. For the superlinear case, he discussed one special case and stated the following conjecture.

CONJECTURE 1.1. *Let $f(x, u)$ be increasing in u for $0 \leq u < \infty$, $0 \leq x \leq 1$, $f(x, u) > 0$ for $u > 0$, and (uniformly in x)*

$$\lim_{u \to 0^+} \frac{f(x, u)}{u} = 0, \qquad \lim_{u \to +\infty} \frac{f(x, u)}{u} = +\infty.$$
(1.2)

Then problem (1.1) has a positive solution.

In this paper, we give a negative answer to this conjecture by constructing a counterexample. Moreover, we give a general sufficient condition to guarantee the existence of positive solutions of (1.1), precisely to say, we modify the equalities in (1.2) a little (see (3.4) in Section 3), and then using topological degree theory and cone theory to prove that (1.1) has at least one positive solution under this modified condition, and that the other conditions in the conjecture are unnecessary.

2. Some lemmas. In what follows, denote by $\|\cdot\|$ the supremum norm of the Banach space $C[0,1]$. Let $P = \{u \in C[0,1] \mid u(x) \geq 0 \text{ for } x \in [0,1]\}$, then P is a cone of $C[0,1]$ and it induces a partial ordering in $C[0,1]$, that is, $u \leq v$ if and only if $u(x) \leq v(x)$ for $x \in [0,1]$ (cf. [1, 2]). Denote by $U_l = \{u \in C[0,1] \mid \|u\| < l\}$ $(l > 0)$, the open ball in $C[0,1]$.

LEMMA 2.1. *For $u \in C^4[0,1]$, u is a solution of ([1.1](#)) if and only if $u \in C[0,1]$ is a solution of the integral equation*

$$u(x) = Au(x) \equiv \xi(x) + Bu(x), \tag{2.1}$$

where

$$\xi(x) = \alpha(1-x)^2(2x+1) + \beta x(1-x)^2 + \gamma x^2(3-2x) + \delta x^2(1-x), \tag{2.2}$$

$$Bu(x) = \int_0^1 G(x,y)f(y,u(y))\,dy, \tag{2.3}$$

where $G(x,y)$ is the Green's function for u'''' with the boundary conditions $u(0) = u'(0) = u(1) = u'(1) = 0$, that is,

$$G(x,y) = \begin{cases} \dfrac{x^2}{6}(1-y)^2(3y-2xy-x), & 0 \le x \le y \le 1, \\[3mm] \dfrac{y^2}{6}(1-x)^2(3x-2xy-y), & 0 \le y \le x \le 1. \end{cases} \tag{2.4}$$

PROOF. The conclusion can be verified in a standard way. □

REMARK 2.2. It is clear that $G(x,y)$ is continuous on $[0,1] \times [0,1]$, $G(x,y) = G(y,x)$ for $0 \le x,\, y \le 1$, and $G(x,y) > 0$ for $0 < x,y < 1$.

LEMMA 2.3. *Let $G(x,y)$ be defined by ([2.4](#)), then*

$$G(x,y) \ge \frac{2}{3}x^2(1-x)^2 G(z,y), \quad \forall x,y,z \in [0,1]. \tag{2.5}$$

PROOF

CASE 1 ($0 \le x \le y \le 1$). (1) If $z \le y$, then

$$G(x,y) = \frac{x^2}{6}(1-y)^2(3y-2xy-x) \ge \frac{x^2}{6}(1-y)^2 \cdot 2y(1-x)$$

$$\ge 2x^2(1-x)\frac{(1-y)^2}{6}y \cdot \frac{z(3y-2yz-z)}{3} \tag{2.6}$$

$$\ge \frac{2x^2(1-x)}{3}G(z,y).$$

(2) If $y \le z$, then

$$G(x,y) = \frac{x^2}{6}(1-y)^2(3y-2xy-x) \ge \frac{2x^2(1-x)}{6}(1-y)^2 y$$

$$\ge \frac{2x^2(1-x)}{6}(1-z)^2 y \cdot \frac{y(3z-2yz-y)}{3} \tag{2.7}$$

$$= \frac{2x^2(1-x)}{3}G(z,y).$$

CASE 2 ($0 \le y \le x \le 1$). It is easy to verify in a similar way as in [Case 1](#)

$$G(x,y) \ge \frac{2x(1-x)^2}{3}G(z,y), \quad \forall x,y,z \in [0,1],\ y \le x. \tag{2.8}$$

Hence,

$$G(x,y) \geq \frac{2x^2(1-x)^2}{3} G(z,y), \quad \forall x,y,z \in [0,1]. \tag{2.9}$$

This completes the proof.   □

LEMMA 2.4. *There exists $0 < \sigma_1 \leq 1$ such that*

$$\xi(x) \geq \frac{2\sigma_1}{3} x^2(1-x)^2 \xi(y), \quad \forall x,y \in [0,1]. \tag{2.10}$$

PROOF. By (2.2), it is easy to see that

$$0 \leq \xi(x) \leq \alpha + \frac{4}{27}\beta + \gamma + \frac{4}{27}\delta \equiv \omega, \quad \forall x \in [0,1]. \tag{2.11}$$

If $\omega = 0$, then $\xi(x) \equiv 0$ and (2.10) is satisfied for any $0 < \sigma_1 \leq 1$.

We now suppose that $\omega > 0$. If $\alpha > 0$, then choose $0 < \sigma_2 \leq \min\{1, 3\alpha/2\omega\}$, we have

$$\begin{aligned}
\xi(x) &\geq \alpha(1-x)^2(2x+1) \geq \alpha(1-x)^2 \geq \frac{2}{3}\omega\sigma_2(1-x)^2 \\
&\geq \sigma_2 \cdot \frac{2}{3} x^2(1-x)^2 \omega \geq \frac{2\sigma_2}{3} x^2(1-x)^2 \xi(y), \quad \forall x,y \in [0,1].
\end{aligned} \tag{2.12}$$

If $\alpha = 0$, $\beta > 0$ (or $\gamma > 0$, or $\delta > 0$), the proof is similar. This completes the proof.   □

REMARK 2.5. Set

$$v^*(x) = \frac{2\sigma_1}{3} x^2(1-x)^2 \equiv \sigma x^2(1-x)^2, \tag{2.13}$$

where σ_1 is given by (2.10). Then (2.5) remains valid if we replace $(2/3)x^2(1-x)^2$ by $v^*(x)$.

LEMMA 2.6. *For any $u \in P$, there exists $\tau > 0$ such that*

$$v^*(x) \geq \tau Bu(x), \quad \forall x \in [0,1]. \tag{2.14}$$

In particular, let $f \equiv 1$ in (2.3), then for some $\tau_0 > 0$,

$$v^*(x) \geq \tau_0 \int_0^1 G(x,y)\,dy, \quad \forall x \in [0,1]. \tag{2.15}$$

PROOF. To prove (2.14) it suffices to prove that there exists $\tau > 0$ such that

$$p(x) \equiv x^2(1-x)^2 \geq \tau \int_0^1 G(x,y)f(y,u(y))\,dy = \tau Bu(x), \quad \forall x \in [0,1]. \tag{2.16}$$

Since

$$G(0,y) = 0, \quad \frac{\partial G(0,y)}{\partial x} = 0, \quad \frac{\partial^2 G(0,y)}{\partial x^2} = y(1-y)^2, \quad \forall y \in [0,1], \tag{2.17}$$

we have

$$Bu(0) = (Bu)'(0) = 0, \quad (Bu)''(0) = \int_0^1 y(1-y)^2 f(y,u(y))\,dy \equiv \tau_1 \geq 0. \tag{2.18}$$

On the other hand, $p(0) = p'(0) = 0$, $p''(0) = 2 > 0$. Hence, there exists $\tau_2 > 0$ such that $p''(0) > \tau_2 (Bu)''(0)$. Thus, for some small $\delta_1 > 0$,

$$p(x) \geq \tau_2 Bu(x), \quad \forall 0 \leq x \leq \delta_1. \tag{2.19}$$

Similarly, there exist $\tau_3 > 0$ and $\delta_2 > 0$ sufficiently small such that

$$p(x) \geq \tau_3 Bu(x), \quad \forall 1 - \delta_2 \leq x \leq 1. \tag{2.20}$$

Therefore, there exists $\tau > 0$ such that (2.16) holds. So, Lemma 2.6 is proved.   □

Consider the linear eigenvalue problem

$$u'''' = \lambda u, \quad 0 < x < 1, \quad u(0) = u'(0) = u(1) = u'(1) = 0. \tag{2.21}$$

It is easy to see that (2.21) is equivalent to the following integral equation:

$$u(x) = \lambda \int_0^1 G(x,y)u(y)\,dy \equiv \lambda Lu(x), \tag{2.22}$$

where $G(x,y)$ is given by (2.4), and it is easy to verify by $G(x,y) > 0$ ($x,y \in (0,1)$) that the spectral radius of L, $r(L)$ is positive, therefore Krein-Rutman's theorem (see [4]) shows that there exists $\phi(x) \in P$, $\phi(x) \not\equiv 0$ and $\|\phi\| = 1$ such that

$$\phi(x) = \lambda_1 \int_0^1 G(x,y)\phi(y)\,dy, \tag{2.23}$$

where $\lambda_1 = r(L)^{-1} > 0$ is the first eigenvalue of (2.21).

LEMMA 2.7. *There exists $c \in (1/3, 2/3)$ such that ϕ is strictly increasing on $[0,c]$ and strictly decreasing on $[c,1]$. Therefore*

$$\phi(x) \geq \min\left\{\phi\left(\frac{1}{3}\right), \phi\left(\frac{2}{3}\right)\right\} \equiv \phi_0 > 0, \quad \forall \frac{1}{3} < x < \frac{2}{3}. \tag{2.24}$$

PROOF. By (2.23), we have

$$\phi(0) = \phi(1) = 0; \quad \phi(x) > 0, \quad \forall 0 < x < 1. \tag{2.25}$$

By differentiating (2.23) we get

$$\phi'(x) = \lambda_1 \int_0^x \frac{y^2}{2}(x-1)(3x - 2xy - 1)\phi(y)\,dy$$
$$+ \lambda_1 \int_x^1 \frac{x}{2}(y-1)^2(2y - 2xy - x)\phi(y)\,dy \equiv p(x) + q(x). \tag{2.26}$$

Clearly, $p(0) = p(1) = 0$, $q(0) = q(1) = 0$. An elementary computation shows that

$$p(x) > 0, \quad \forall 0 < x \leq \frac{1}{3}; \qquad p(x) < 0, \quad \forall \frac{1}{2} < x < 1;$$
$$q(x) > 0, \quad \forall 0 < x < \frac{1}{2}; \qquad q(x) < 0, \quad \forall \frac{2}{3} < x < 1. \tag{2.27}$$

Differentiating (2.23) twice, we get

$$\phi''(x) = \lambda_1 \int_0^x y^2(3x - 2xy + y - 2)\phi(y)\,dy$$
$$+ \lambda_1 \int_x^1 (y-1)^2(y - 2xy - x)\phi(y)\,dy \equiv p_1(x) + q_1(x). \tag{2.28}$$

It is easy to verify that

$$p_1(x) < 0, \quad \forall 0 < x \le \frac{2}{3}; \qquad q_1(x) < 0, \quad \forall \frac{1}{3} \le x < 1. \tag{2.29}$$

Hence $\phi''(x) < 0$ for $1/3 \le x \le 2/3$, that is, $\phi'(x)$ is strictly decreasing on $[1/3, 2/3]$. By the above discussion we know that there exists $c \in (1/3, 2/3)$ such that

$$\phi'(x) > 0, \quad \forall 0 < x < c; \qquad \phi'(x) < 0, \quad \forall c < x < 1. \tag{2.30}$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

LEMMA 2.8 (see [5]). *Let P be a solid cone in E, $A : P \to P$ a completely continuous increasing operator. Suppose that*
   (i) *there exist $u_0, v_0 \in P$ such that $u_0 \le Au_0$, $Av_0 \le v_0$, $u_0 \not\le v_0$,*
   (ii) *there exist $u^* \in P$ and $\delta > 0$ such that $Au \ge \delta \|Au\| u^*$ for all $u \in P$,*
   (iii) *$v_0 \in \mathrm{Int}\,P$, and there exists $d > 0$ such that $u^* \ge du_0$.*
   Then A has a nonzero fixed point in P.

3. Main results. We first give a negative answer to Korman's conjecture by constructing a counterexample.

COUNTEREXAMPLE. *Let $f(x, u) = u^2$ for all $x \in [0, 1]$, $0 \le u < +\infty$. If $\alpha \ge 2\lambda_1$, $\gamma \ge 2\lambda_1$, where λ_1 is given by (2.23), then (1.1) has no positive solutions.*

PROOF. By $\alpha \ge 2\lambda_1$, $\gamma \ge 2\lambda_1$ and (2.2) we have

$$\xi(x) \ge \alpha(1-x)^2 + \gamma x^2 \ge 2\lambda_1(1 - 2x + 2x^2) \ge \lambda_1, \quad \forall x \in [0, 1]. \tag{3.1}$$

If u is a positive solution of (1.1), that is, $u = Au$, then

$$u(x) = Au(x) = \xi(x) + Bu(x) \ge \xi(x) \ge \lambda_1, \quad \forall x \in [0, 1]. \tag{3.2}$$

So by (2.23) and Remark 2.2, we have

$$\int_0^1 \phi(x)u(x)\,dx = \int_0^1 \phi(x)\xi(x)\,dx + \int_0^1 \phi(x)\,dx \int_0^1 G(x,y)u^2(y)\,dy$$
$$= \int_0^1 \phi(x)\xi(x)\,dx + \int_0^1 u^2(y)\,dy \int_0^1 G(y,x)\phi(x)\,dx$$
$$\ge \int_0^1 \phi(x)\xi(x)\,dx + \int_0^1 \lambda_1 u(y)\,dy \int_0^1 G(y,x)\phi(x)\,dx$$
$$\ge \lambda_1 \int_0^1 \phi(x)\,dx + \int_0^1 \phi(y)u(y)\,dy. \tag{3.3}$$

This is a contradiction. Hence $u \neq Au$ for all $u \in P$, that is, (1.1) has no positive solutions. This completes the proof. $\qquad\square$

Clearly, $f(x,u) = u^2$ satisfies all the conditions of the conjecture. So counterexample shows that Korman's conjecture is false. In what follows, we will investigate the existence of positive solutions of (1.1) by modifying the conditions of the conjecture. The following conditions will play roles in this section

$$\liminf_{u\to+\infty}\min_{x\in[0,1]}\frac{f(x,u)}{u} > \lambda_1, \qquad \limsup_{u\to0^+}\max_{x\in[0,1]}\frac{f(x,u+\xi(x))}{u} < \lambda_1, \tag{3.4}$$

$$\limsup_{u\to+\infty}\max_{x\in[0,1]}\frac{f(x,u)}{u} < \lambda_1, \qquad \liminf_{u\to0^+}\min_{x\in[0,1]}\frac{f(x,u)}{u} > \lambda_1, \tag{3.5}$$

where λ_1 is given by (2.23).

$$\sup_{u>0}\min_{x\in[0,1]}\frac{f(x,u)}{u} = +\infty, \qquad \inf_{u\geq2\|\xi\|}\max_{x\in[0,1]}\frac{f(x,u)}{u} = 0. \tag{3.6}$$

THEOREM 3.1. *Let (3.4) be satisfied. If $\xi(x) \equiv 0$ ($x \in [0,1]$), then BVP (1.1) has at least one positive solution; if $\xi(x) \not\equiv 0(x \in [0,1])$, then BVP (1.1) has at least two positive solutions.*

PROOF. Suppose that $v^*(x)$ is defined by (2.13), $\phi(x)$ and λ_1 are given by (2.23), and $\eta_1 \equiv \int_0^1 v^*(x)\phi(x)dx > 0$. Denote

$$Q = \left\{u \in P \ \middle| \ \int_0^1 u(x)\phi(x)\,dx \geq \eta\|u\|\right\}, \tag{3.7}$$

where $\eta = \min\{\eta_1, \int_0^1 \phi(x)dx\} > 0$. It is easy to see that $Q \neq \varnothing, Q \neq \{\theta\}$ is also a cone in $C[0,1]$.

We now show that for the operator A defined by (2.1),

$$A(P) \subset Q. \tag{3.8}$$

In fact, by Lemma 2.4,

$$\int_0^1 \xi(x)\phi(x)\,dx \geq \int_0^1 v^*(x)\phi(x)\,dx \cdot \xi(y) \geq \eta \cdot \xi(y), \quad \forall y \in [0,1], \tag{3.9}$$

that is, $\xi \in Q$.

By (2.23), Remark 2.2, (2.5), and Remark 2.5,

$$\phi(x) = \lambda_1 \int_0^1 G(x,y)\phi(y)\,dy = \lambda_1 \int_0^1 G(y,x)\phi(y)\,dy$$

$$\geq \lambda_1 \int_0^1 v^*(y)G(z,x)\phi(y)\,dy = \lambda_1\eta_1 G(z,x), \quad \forall z,x \in [0,1]. \tag{3.10}$$

Hence for $u \in P$,

$$\int_0^1 \phi(x) Bu(x)\, dx$$
$$= \int_0^1 \phi(x)\, dx \int_0^1 G(x,y) f(y,u(y))\, dy = \int_0^1 f(y,u(y))\, dy \int_0^1 G(y,x)\phi(x)\, dx$$
$$= \frac{1}{\lambda_1} \int_0^1 \phi(y) f(y,u(y))\, dy \geq \frac{\lambda_1 \eta_1}{\lambda_1} \int_0^1 G(z,y) f(y,u(y))\, dy$$
$$\geq \eta Bu(z), \quad \forall z \in [0,1],$$
$$(3.11)$$

that is, $Bu \in Q$. Therefore $Au = \xi + Bu \in Q$.

By (3.4), there exist $R_1 > \|\xi\|$ and $\varepsilon > 0$ such that

$$f(x,u) \geq (\lambda_1 + \varepsilon) u, \quad \forall x \in [0,1],\ u \geq R_1. \tag{3.12}$$

Hence, there exists a constant $C = C(R_1) > 0$ such that

$$f(x,u) \geq (\lambda_1 + \varepsilon) u - C, \quad \forall x \in [0,1],\ u \geq 0. \tag{3.13}$$

Choose $u_0 \in Q \setminus \{\theta\}$, we assert that for any fixed $R > \max\{R_1, \|\xi\| + 1, C/(\eta\varepsilon)\} > 0$,

$$u - Au \neq \lambda u_0, \quad \forall u \in P,\ \|u\| = R,\ \lambda \geq 0. \tag{3.14}$$

Otherwise, suppose that there exist $u_1 \in P$, $\|u_1\| = R$ and $\lambda_0 \geq 0$ such that $u_1 - Au_1 = \lambda_0 u_0$, then by (3.13), Remark 2.2, and (2.23) we have

$$0 \geq -\lambda_0 \int_0^1 u_0(x)\phi(x)\, dx = \int_0^1 \phi(x) Au_1(x)\, dx - \int_0^1 \phi(x) u_1(x)\, dx$$
$$\geq \int_0^1 \phi(x) Bu_1(x)\, dx - \int_0^1 \phi(x) u_1(x)\, dx$$
$$\geq \int_0^1 \phi(x)\, dx \int_0^1 G(x,y) \big[(\lambda_1 + \varepsilon) u_1(y) - C\big]\, dy - \int_0^1 \phi(x) u_1(x)\, dx$$
$$= \int_0^1 \big[(\lambda_1 + \varepsilon) u_1(y) - C\big]\, dy \int_0^1 G(y,x)\phi(x)\, dx - \int_0^1 \phi(x) u_1(x)\, dx$$
$$= \frac{\lambda_1 + \varepsilon}{\lambda_1} \int_0^1 u_1(y)\phi(y)\, dy - \frac{C}{\lambda_1} \int_0^1 \phi(y)\, dy - \int_0^1 \phi(x) u_1(x)\, dx$$
$$= \frac{\varepsilon}{\lambda_1} \int_0^1 u_1(y)\phi(y)\, dy - \frac{C}{\lambda_1} \int_0^1 \phi(y)\, dy.$$
$$(3.15)$$

Observing $u_1 = Au_1 + \lambda_0 u_0 \in Q$, we have

$$\frac{C}{\varepsilon} \geq \frac{C}{\varepsilon} \int_0^1 \phi(y)\, dy \geq \int_0^1 u_1(y)\phi(y)\, dy \geq \eta\|u_1\| = \eta R, \tag{3.16}$$

in contradiction with $R > C/(\varepsilon\eta)$. Hence (3.14) holds.

Denote
$$P_\xi = \{u(x) + \xi(x) \mid u \in P\}, \quad \Omega_R = P_\xi \cap U_R. \tag{3.17}$$

Then $A : P \to P_\xi$ is completely continuous. Without loss of generality, suppose that $Au \neq u$ for $u \in \partial U_R \cap P_\xi$. Therefore the fixed point index $i(A, \Omega_R, P_\xi)$ is well defined (see [1, 2]). We now prove that

$$i(A, \Omega_R, P_\xi) = 0. \tag{3.18}$$

Let u_0 be given as above, denote $a = \|u_0\|$, $b = \sup_{\bar{\Omega}_R} \|Au\|$. Choose $\rho > 0$ sufficiently large such that $\rho a > b + R$. Then it follows from (3.14) and the homotopy invariance of the fixed point index that

$$i(A, \Omega_R, P_\xi) = i(A + \rho u_0, \Omega_R, P_\xi). \tag{3.19}$$

The latter equals 0 since for any $u \in \bar{\Omega}_R$

$$\|Au + \rho u_0\| \geq \rho \|u_0\| - \|Au\| \geq \rho a - b > R \geq \|u\|. \tag{3.20}$$

Thus (3.18) holds.

On the other hand, by (3.4), there exist $0 < r < \min\{R_1 - \|\xi\|, 1\}$ and $\varepsilon_1 > 0$ sufficiently small such that

$$f(x, u + \xi(x)) \leq (\lambda_1 - \varepsilon_1)u, \quad \forall x \in [0,1], \ 0 \leq u \leq r. \tag{3.21}$$

We now prove that

$$\xi + tBu \neq u, \quad \forall t \in [0,1], \ u \in P_\xi, \ \|u - \xi\| = r. \tag{3.22}$$

In fact, if $t_1 \in [0,1]$, $u_2 \in P_\xi$, $\|u_2 - \xi\| = r$ satisfy $\xi + t_1 Bu_2 = u_2$, then by (2.23), and (3.21) we have

$$\int_0^1 \phi(x) u_2(x)\, dx$$
$$= t_1 \int_0^1 \phi(x)\, dx \int_0^1 G(x,y) f(y, u_2(y))\, dy + \int_0^1 \phi(x)\xi(x)\, dx$$
$$\leq \int_0^1 \phi(x)\, dx \int_0^1 G(x,y) f(y, (u_2(y) - \xi(y)) + \xi(y))\, dy + \int_0^1 \phi(x)\xi(x)\, dx$$
$$\leq (\lambda_1 - \varepsilon_1) \int_0^1 (u_2(y) - \xi(y))\, dy \int_0^1 G(x,y)\phi(x)\, dx + \int_0^1 \phi(x)\xi(x)\, dx$$
$$= \frac{\lambda_1 - \varepsilon_1}{\lambda_1} \int_0^1 (u_2(y) - \xi(y))\phi(y)\, dy + \int_0^1 \phi(x)\xi(x)\, dx < \int_0^1 \phi(y) u_2(y)\, dy, \tag{3.23}$$

a contradiction. Therefore (3.22) holds. By the homotopy invariance of the fixed point index, we have

$$i(A, \Omega_r, P_\xi) = i(\xi, \Omega_r, P_\xi) = 1, \tag{3.24}$$

where $\Omega_r = P_\xi \cap \{u \mid \|u - \xi\| < r\}$. Obviously, $\bar{\Omega}_r \subset \Omega_R$.

By (3.18), (3.24), and the additivity of the fixed point index we get

$$i(A, \Omega_R \setminus \bar{\Omega}_r, P_\xi) = -1, \tag{3.25}$$

which means that A has a fixed point in $\Omega_R \setminus \bar{\Omega}_r$. Also, by (3.24), A has a fixed point in Ω_r.

Consequently, if $\xi(x) \not\equiv 0$ ($x \in [0,1]$), that is, $\theta \notin P_\xi$, then A has at least two fixed points in P_ξ, that is, (1.1) has at least two positive solutions; if $\xi(x) \equiv 0$ ($x \in [0,1]$), then A has at least one nonzero fixed point in P_ξ, that is, (1.1) has at least one positive solution. This completes the proof. $\qquad\square$

In a similar way as proving Theorem 3.1, we obtain the following theorem.

THEOREM 3.2. *If (3.5) is satisfied, then BVP (1.1) has at least one positive solution.*

REMARK 3.3. Theorems 3.1 and 3.2 cover the superlinear and the sublinear nonlinearities, respectively. As is well known that conditions (3.4) and (3.5) expressed by λ_1 are more general than those expressed by 0 and $+\infty$, to say, Theorem 3.2 improves [3, Theorem 3]. Furthermore, it is clear that Theorem 3.1 remains valid if we replace the second inequality in (3.4) by

$$\limsup_{u \to 0^+} \max_{0 \le x \le 1, 0 \le v \le \|\xi\|} \frac{f(x, u+v)}{u} < \lambda_1. \tag{3.26}$$

It is easy to give some simple functions f, which satisfy (3.4) and the above inequality. For example

$$f(x, u) = \begin{cases} 0, & 0 \le u \le \|\xi\|, \\ \left(u - \|\xi\|\right)^2, & u > \|\xi\|. \end{cases} \tag{3.27}$$

THEOREM 3.4. *Let $f(x, u)$ be increasing in u. If (3.6) is satisfied, then BVP (1.1) has at least one positive solution.*

PROOF. Let P be the cone of $C[0,1]$ defined in Section 2, then P is normal and solid (see [1, 2]). Let A and B be defined by (2.1) and (2.3), respectively, then $A, B : P \to P$ are completely continuous operators. Also, A and B are increasing since $f(x, u)$ is increasing in u.

By condition (3.6), there exists $\bar{u} > 0$ such that

$$f(x, \bar{u}) > \frac{\lambda_1}{\phi_0 \tau_0 g_0} \bar{u}, \quad \forall x \in [0,1], \tag{3.28}$$

where ϕ_0 is given by (2.24), τ_0 is given by (2.15) and $g_0 = \int_{1/3}^{2/3} G(1/2, y) \, dy > 0$. We assert that

$$\frac{\bar{u}}{\phi_0} \phi(x) \le B\left(\frac{\bar{u}}{\phi_0} \phi(x)\right), \tag{3.29}$$

where ϕ is given by (2.23). In fact, by (2.5), (2.15), (2.23), (2.24), and (3.28) we have

$$
\begin{aligned}
B\left(\frac{\bar{u}}{\phi_0}\phi(x)\right) &= \int_0^1 G(x,y)f\left(y,\frac{\bar{u}}{\phi_0}\phi(y)\right)dy \geq \int_{1/3}^{2/3} G(x,y)f\left(y,\frac{\bar{u}}{\phi_0}\phi(y)\right)dy \\
&\geq \int_{1/3}^{2/3} G(x,y)f(y,\bar{u})\,dy \geq \frac{\lambda_1}{\phi_0\tau_0 g_0}\int_{1/3}^{2/3} G(x,y)\,dy\cdot\bar{u} \\
&\geq \frac{\lambda_1\bar{u}}{\phi_0\tau_0 g_0}\int_{1/3}^{2/3} v^*(x)G(z,y)\,dy \geq \frac{\lambda_1\bar{u}}{\phi_0 g_0}\int_{1/3}^{2/3} G(z,y)\,dy\int_0^1 G(x,t)\,dt \\
&\geq \frac{\lambda_1\bar{u}}{\phi_0 g_0}\int_{1/3}^{2/3} G(z,y)\,dy\int_0^1 G(x,t)\phi(t)\,dt = \frac{\bar{u}}{\phi_0 g_0}\int_{1/3}^{2/3} G(z,y)\,dy\cdot\phi(x) \\
&= \frac{1}{g_0}\int_{1/3}^{2/3} G(z,y)\,dy\cdot\frac{\bar{u}}{\phi_0}\phi(x), \quad \forall x,z\in[0,1].
\end{aligned}
\tag{3.30}
$$

Hence (3.29) holds. Denote $u_0(x) = B((\bar{u}/\phi_0)\phi(x))$, then

$$
Au_0(x) = \xi(x) + Bu_0(x) \geq B\left(B\left(\frac{\bar{u}}{\phi_0}\phi(x)\right)\right) \geq B\left(\frac{\bar{u}}{\phi_0}\phi(x)\right) = u_0(x), \tag{3.31}
$$

that is, $u_0(x)$ is a subsolution of A.

On the other hand, by (3.6), there exists a constant $\bar{v} > 2\|\xi\|$ such that

$$
f(x,\bar{v}) < \frac{\bar{v}}{2g_1}, \quad \forall x\in[0,1], \tag{3.32}
$$

where $g_1 = \max\limits_{x\in[0,1]}\int_0^1 G(x,y)\,dy$. Then

$$
A\bar{v}(x) = \xi(x) + B\bar{v}(x) \leq \xi(x) + \int_0^1 G(x,y)\,dy\cdot\frac{\bar{v}}{2g_1} \leq \xi(x) + \frac{\bar{v}}{2} \leq \bar{v}, \tag{3.33}
$$

that is, $\bar{v}$ is a supersolution of A.

(i) If $u_0 \leq \bar{v}$, then it follows from (3.31), (3.33), and [1, Theorem 19.1] that A has a fixed point $w^*(x)$ satisfying $u_0 \leq w^* \leq \bar{v}$.

(ii) If $u_0 \nleq \bar{v}$, then define

$$
P(v^*) = \{u\in P \mid u(x) \geq \|u\|v^*(x),\ \forall x\in[0,1]\}, \tag{3.34}
$$

where v^* is given by (2.13). We show that A maps P into $P(v^*)$. In fact, for any $u\in P$, we have by (2.5)

$$
\begin{aligned}
Bu(x) &= \int_0^1 G(x,y)f(y,u(y))\,dy \\
&\geq \int_0^1 v^*(x)G(z,y)f(y,u(y))\,dy = v^*(x)Bu(z), \quad \forall x,z\in[0,1].
\end{aligned}
\tag{3.35}
$$

Hence by Lemma 2.4 and Remark 2.5, we have

$$Au(x) = \xi(x) + Bu(x) \geq v^*(x)\xi(y) + v^*(x)Bu(y) = v^*(x)Au(y), \quad \forall x, y \in [0,1].$$
(3.36)

Thus $Au(x) \geq \|Au\|v^*(x)$ ($\forall x \in [0,1]$), that is, Lemma 2.8(ii) is satisfied for $u^*(x) = v^*(x)$, $\delta = 1$. Moreover, $\bar{v} \in \mathrm{Int}\,P$, and by (2.14), $v^*(x) \geq \tau B((\bar{u}/\phi_0)\phi(x)) = \tau u_0(x)$ for some $\tau > 0$, that is, Lemma 2.8(iii) holds. Therefore, by (3.31), (3.33), and Lemma 2.8, A has a nonzero fixed point in P. This completes the proof. $\qquad\square$

REMARK 3.5. In a similar way as establishing Lemmas 2.3, 2.6, 2.7 and Theorems 3.1, 3.2 and 3.4, one can also discuss the existence of positive solutions for other physically important boundary conditions for $Lu = u''''$.

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