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BIHARMONIC MAPS ON V-MANIFOLDS

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Abstract. We generalize biharmonic maps between Riemannian manifolds into the case of the domain being V-manifolds. We obtain the first and second variations of biharmonic maps on V-manifolds. Since a biharmonic map from a compact V-manifold into a Riemannian manifold of nonpositive curvature is harmonic, we construct a biharmonic non-harmonic map into a sphere. We also show that under certain condition the biharmonic property of f implies the harmonic property of f. We finally discuss the composition of biharmonic maps on V-manifolds.

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1. Introduction. Following Eells, Sampson, and Lemaire's tentative ideas [7, 8, 9], Jiang first discussed biharmonic (or 2-harmonic) maps between Riemannian manifolds in his two articles [10, 11] in China in 1986, which gives the conditions for biharmonic maps. A biharmonic map $f : M \to N$ between Riemannian manifolds is the smooth critical point of the bi-energy functional

$$E_2(f) = \int_M \|(d+d^*)f\|^2 * 1 = \int_M \|\tau(f)\|^2 * 1, \tag{1.1}$$

where $*1$ is the volume form on M, the tension field $\tau(f) = (\hat{D}df)(e_i, e_i) (= (\hat{D}_{e_i}df)(e_i))$, $\{e_i\}$ is the local frame of a point p in M. Biharmonic maps are the extensions of harmonic maps, and their study provides a source in partial differential equations, differential geometry, and analysis. After Jiang, Chiang, and Sun have studied biharmonic maps in two papers [6, 14]. Chiang also studied harmonic maps and biharmonic maps of two different kinds of singular spaces: V-manifolds [3, 4] and spaces with conical singularities (with Andrea Ratto [5]).

In this paper, we generalize the notion of a biharmonic map to the case that the domain of f is a V-manifold due to Satake in [1, 12, 13]. A (C^∞) V-manifold $(M, \mathcal{F})$ consists of a Hausdorff space M with an atlas $\mathcal{F}$ of V-charts satisfying the following conditions:

(i) If $\{\tilde{U}, G, \pi\}$ and $\{\tilde{U}', G', \pi'\}$ are two V-charts in $\mathcal{F}$ over U, U', respectively, in M with $U \subset U'$, then there exists an injection $\lambda : \{U, G, \pi\} \to \{U, G', \pi'\}$.

(ii) The supports of V-charts in $\mathcal{F}$ form a basis for open sets in M.

Take a chart $\{\tilde{U}, G, \pi\} \in \mathcal{F}$ such that $p \in \pi(\tilde{U})$ and choose $\tilde{p} \in \tilde{U}$ such that $\sigma\tilde{p} = \tilde{p}$. The isotropic subgroup $G_{\tilde{p}}$ of G at $\tilde{p}$ is the set of all $\sigma \in G$ such that $\sigma\tilde{p} = \tilde{p}$. So $G_{\tilde{p}}$ is called the *isotropic group* of p. The singular set $\mathbb{S}$ of M consists of all singular points of M, that is, the points of M with nontrivial isotropy groups. (For example, S^2/Z_3 is a compact V-manifold with two singular points.) The main difficulties of this paper arise from the complicated behavior of the singular locus of V-manifolds, and therefore a different method than the usual one is required. In fact, this article is the extension of Chiang's previous two papers [3, 4].

We derive the first variations of biharmonic maps in Theorem 2.2, and give the definition for biharmonic maps on V-manifolds. We show that a biharmonic map from a compact V-manifold into a Riemannian manifold of nonpositive curvature is a harmonic map in Theorem 2.4. Then we construct a biharmonic non-harmonic map from a V-manifold into a sphere in Section 2. We obtain the second variations of biharmonic maps in Theorem 3.1. If $d^2/dt^2 E_2(f_t)|_{t=0} \geq 0$, then f is a stable biharmonic map. In Theorem 3.3, we show that if a stable biharmonic map from a compact V-manifold M into a Riemannian manifold N of positive curvature satisfies the conservation law, then f must be a harmonic map. In Theorem 3.4, we prove the composition of biharmonic maps on V-manifolds which generalizes Sun's result in [14].

2. Biharmonic maps on V-manifolds. Let $(M,\mathcal{F})$ be a $(\mathbb{C}^\infty)$ V-manifold, and U be an open subset of M. By a V-chart on M over U we mean a system $\{\tilde{U},G,\pi\}$ consisting of (1) a connected open subset $\tilde{U}$ of $\mathbb{R}^m$, (2) a finite group G of diffeomorphisms of $\tilde{U}$, with the set of fixed points of codimension ≥ 2, and (3) a continuous map of $\pi:\tilde{U}\to U$ such that $\pi\circ\sigma=\pi$ for $\sigma\in G$ and such that π induces a homeomorphism of $\tilde{U}/G$ onto U. The set U is called the *support* of V-chart, and π is called the *projection* onto U.

Let $(M,\mathcal{F})$ be a V-manifold and $p\in M$. Take a chart $\{\tilde{U},G,\pi\}\in\mathcal{F}$ such that $p\in\pi(\tilde{U})$ and choose $\tilde{p}\in\tilde{U}$ such that $\pi(\tilde{p})=p$. The isotropic subgroup $G_{\tilde{p}}$ of G at $\tilde{p}$ is the set of all $\sigma\in G$ such that $\sigma\tilde{p}=\tilde{p}$, and is uniquely determined by p. Therefore, $G_{\tilde{p}}$ is called the *isotropic group* of p. The singular set $\mathbb{S}$ of M consists of all singular points of M, that is, the points of M with nontrivial isotropic groups. Let $(\tilde{x}^1,\dots,\tilde{x}^m)$ be a coordinate system around $\tilde{p}$ and consider the system $\tilde{y}^i=1/|G_{\tilde{p}}|\sum l_{ij}(\sigma^{-1})\tilde{x}^j\cdot\sigma$ with

$$l_{ij}(\sigma)=\left[\frac{\partial\tilde{x}^i\circ\sigma}{\partial\tilde{x}^j}\right]_{\tilde{p}},\qquad |G_{\tilde{p}}|=\text{order of } G_{\tilde{p}}. \tag{2.1}$$

Then the $\{\tilde{y}^i\}$ are a new coordinate system around $\tilde{p}$ and $G_{\tilde{p}}$ operates linearly in the $\tilde{y}$-system. After this suitable C^∞ change of coordinates around $\tilde{p}$, $G_{\tilde{p}}$ becomes a finite group of linear transformations. The fixed point set of any $\sigma\in G_{\tilde{p}}$ is the defined linear equations in the $\tilde{y}$, and consequently the fixed point set of $\sigma\in G_{\tilde{p}}$ in $\tilde{U}$ is the intersection of $\tilde{U}$ with a linear space. Therefore, $\pi^{-1}\mathbb{S}$ is locally expressed by a finite union of linear spaces intersected with $\tilde{U}$. Hence $\mathbb{S}$ is a V-submanifold of codimension ≥ 2 of M. Clearly, $M-\mathbb{S}$ is an ordinary manifold.

We fix a V-manifold M with defining atlas $\mathcal{F}$. A *smooth function* $f:(M,\mathcal{F})\to N$ from M into an ordinary manifold N is defined as follows: for any $\{\tilde{U},G,\pi\}\in\mathcal{F}$ there corresponds an ordinary G-invariant smooth map $f_{\tilde{U}}^G=1/|G|\sum_{\sigma\in G}f_{\tilde{U}}\circ\sigma:\tilde{U}\to N$ such that $f_{\tilde{U}}^G=f\circ\pi$ and $f_{\tilde{U}}^G=f_{\tilde{U}'}^{G'}\circ\lambda$ for any injection $\lambda:\{\tilde{U},G,\pi\}\to\{\tilde{U}',G',\pi'\}$ where $f_{\tilde{U}}:\tilde{U}\to N$ is an ordinary smooth map.

Put a Riemannian metric $g_{\tilde{U}}=g_{ij}\,d\tilde{x}^i\,d\tilde{x}^j$ on $\tilde{U}$. By taking the G-average if necessary, we can assume that $g_{\tilde{U}}$ is G-invariant. Thus the transformations $\sigma\in G$ are isometries for $g_{\tilde{U}}$. By using the standard partition of unity construction, we can patch all such local invariant metrics together into a global metric tensor field of type $(0,2)$ on the V-manifold M, which we call a Riemannian metric on M.

Let M^m be a compact V-manifold of dimension m with $\mathbb{C}^\infty$ Riemannian metric g, and N^n a ($\mathbb{C}^\infty$) Riemannian manifold of dimension n. By Satake [12, 13], M admits a finite triangulation $T = \cup s_\alpha$ such that each s_α is contained in the support U_α of a V-chart $\{\tilde{U}_\alpha, G_\alpha, \pi_\alpha\} \in \mathcal{F}$ on M and is the homeomorphic projection of a regular simplex $\tilde{s}_\alpha$ in $\tilde{U}_\alpha$. For a smooth map $f : M \to N$, the bi-energy functional of f is defined by

$$E_2(f) = \int_M |\tau(f)|^2 * 1 = \sum \int_{s_\alpha} |\tau(f)|^2 dx_\alpha = \sum \frac{1}{|G_\alpha|} \int_{\tilde{s}_\alpha} |\tau(\tilde{f})|^2 d\tilde{x}_\alpha, \tag{2.2}$$

where $d\tilde{x}_\alpha$ denotes the volume form with respect to the G_α-invariant metric g_{ij} in $\tilde{U}_\alpha$, $\tilde{f}_\alpha : \tilde{U}_\alpha \to N$ is the G_α -invariant lift of f. The Green's divergence theorem on a compact V-manifold proved in [3] plays an important role in the proofs of both Theorems 2.2 and 3.1.

In order to compute the Euler-Lagrange equation, we consider a one-parameter family of maps $\{f_t\} \in \mathbb{C}^\infty(M,N)$, $t \in I_\epsilon = (-\epsilon, \epsilon)$, $\epsilon > 0$ such that in the V-chart $\{\tilde{U}, G, \pi\} \in \mathcal{F}$ over the support U on M, the G-invariant lift $\tilde{f}_t$ is the endpoint of the segment starting at G-invariant lift $\widetilde{f(x)}$ determined in length and direction by the vector field $\dot{\tilde{f}}$ along $\tilde{f}$, and such that $\partial \tilde{f}_t / \partial t = 0$ and $\bar{D}_{\tilde{e}_i} \partial \tilde{f}_t / \partial t = 0$ outside a compact subset of the interior of $\tilde{U}$. Choose $\{e_i\}$ being the local frame of a point p in U on M, and $\{\tilde{e}_i\}$ being the local frame of the lifting point $\tilde{p}$ in $\tilde{U}$. Let D, D', $\bar{D}$, $\hat{D}$ be the Riemannian connections along TM, TN, $f^{-1}TN$, $T^*M \otimes f^{-1}TN$, and $\tilde{D}$, $\tilde{\hat{D}}$ are the Riemannian connections along $T\tilde{U}$, $T^*\tilde{U} \otimes f^{-1}TN$ in each $\{\tilde{U}, G, \pi\} \in \mathcal{F}$ over the support U on M. Also, let $\triangle = \bar{D}_{\tilde{e}_k} \bar{D}_{\tilde{e}_k} - \bar{D}_{D_{\tilde{e}_k} \tilde{e}_k}$ be the Laplace operator along the cross section of $f^{-1}TN$ in each $\tilde{U}$, and $V = \partial \tilde{f}_t / \partial t$. We can compute (2.2) directly, and obtain the following result.

Lemma 2.1.

$$\begin{aligned} \frac{d}{dt} E_2(f_t) = {} & 2\Sigma \frac{1}{|G_\alpha|} \int_{\tilde{s}_\alpha} \left\langle \tilde{\hat{D}}_{\tilde{e}_i} \tilde{\hat{D}}_{\tilde{e}_i} d\tilde{f}_t\left(\frac{\partial}{\partial t}\right) - \tilde{\hat{D}}_{\tilde{D}_{\tilde{e}_i}\tilde{e}_i} d\tilde{f}_t\left(\frac{\partial}{\partial t}\right), (\tilde{\hat{D}}_{\tilde{e}_j} d\tilde{f}_t)(\tilde{e}_j) \right\rangle d\tilde{x}_\alpha \\ & + 2\Sigma \frac{1}{|G_\alpha|} \int_{\tilde{s}_\alpha} \left\langle R^N\left(d\tilde{f}_t(\tilde{e}_i), d\tilde{f}_t\left(\frac{\partial}{\partial t}\right)\right) d\tilde{f}_t(\tilde{e}_i), (\tilde{\hat{D}}_{\tilde{e}_j} d\tilde{f}_t)(\tilde{e}_j) \right\rangle d\tilde{x}_\alpha. \end{aligned} \tag{2.3}$$

Theorem 2.2. *Let $f : (M, \mathcal{F}) \to N$ be a smooth map from a compact V-manifold $(M, \mathcal{F})$ into a Riemannian manifold N. Set $V = \partial \tilde{f}_t / \partial t$ then*

$$\left.\frac{d}{dt}\right|_{t=0} E_2(f_t) = 2\Sigma \frac{1}{|G_\alpha|} \int_{\tilde{s}_\alpha} \left\langle V, \triangle \tau(\tilde{f}) + R^N(d\tilde{f}(\tilde{e}_i), \tau(\tilde{f})) d\tilde{f}(\tilde{e}_i) \right\rangle d\tilde{x}_\alpha. \tag{2.4}$$

Proof. For every $t \in I_\epsilon$, let

$$\tilde{X} = \left\langle \tilde{\hat{D}}_{\tilde{e}_i} d\tilde{f}_t\left(\frac{\partial}{\partial t}\right), \tilde{\hat{D}}_{\tilde{e}_j} d\tilde{f}_t(\tilde{e}_j) \right\rangle \tilde{e}_i, \qquad \tilde{Y} = \left\langle d\tilde{f}_t\left(\frac{\partial}{\partial t}\right), \bar{D}_{\tilde{e}_i}(\tilde{\hat{D}}_{\tilde{e}_j} d\tilde{f}_t)(\tilde{e}_j) \right\rangle (\tilde{e}_i), \tag{2.5}$$

in each $\{\tilde{U}, \pi, G\} \in \mathcal{F}$ over the support U on M. By computing the divergence of $\tilde{X}$ and $\tilde{Y}$ in each $\tilde{U}$, and applying Green's divergence theorem to the vector field $\tilde{X} - \tilde{Y}$

in each $\tilde{\triangle}$ on the compact manifold M in [3], we have

$$\begin{aligned}\sum \frac{1}{|G_\alpha|}\int_{\tilde{s}_\alpha}&\Big\langle\big(\tilde{\tilde{D}}_{\tilde{e}_i}\tilde{\tilde{D}}_{\tilde{e}_i}d\tilde{f}_t\big)\Big(\frac{\partial}{\partial t}\Big)-\big(\tilde{\tilde{D}}_{\tilde{D}_{\tilde{e}_i}\tilde{e}_i}d\tilde{f}_t\big)\Big(\frac{\partial}{\partial t}\Big),\big(\tilde{\tilde{D}}_{\tilde{e}_j}d\tilde{f}_t\big)(\tilde{e}_j)\Big\rangle d\tilde{x}_\alpha\\ &=\sum\frac{1}{|G_\alpha|}\int_{\tilde{s}_\alpha}\Big\langle d\tilde{f}_t\Big(\frac{\partial}{\partial t}\Big),\bar{D}_{\tilde{e}_k}\bar{D}_{\tilde{e}_k}\big(\tilde{\tilde{D}}_{\tilde{e}_j}d\tilde{f}_t\big)(\tilde{e}_j)-\bar{D}_{\tilde{D}_{\tilde{e}_k}\tilde{e}_k}\Big(\big(\tilde{\tilde{D}}_{\tilde{e}_j}d\tilde{f}_t\big)(\tilde{e}_j)\Big)\Big\rangle d\tilde{x}_\alpha.\end{aligned}\tag{2.6}$$

By the assumption, $\partial\tilde{f}_t/\partial t=0$ and $\bar{D}_{\tilde{e}_i}\partial\tilde{f}_t/\partial t=0$ outside of the compact subset of the interior of each $\tilde{U}$, and substituting (2.6) into (2.3), we get

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}E_2(f_t)=2\sum\frac{1}{|G_\alpha|}\int_{\tilde{s}_\alpha}&\Big\langle d\tilde{f}_t\Big(\frac{\partial}{\partial t}\Big),\bar{D}_{\tilde{e}_k}\bar{D}_{\tilde{e}_k}\big(\tilde{\tilde{D}}_{\tilde{e}_j}d\tilde{f}_t\big)(\tilde{e}_j)\\ &\qquad-\bar{D}_{\tilde{D}_{\tilde{e}_k}\tilde{e}_k}\Big(\big(\tilde{\tilde{D}}_{\tilde{e}_j}d\tilde{f}_t\big)(\tilde{e}_j)\Big)\Big\rangle d\tilde{x}_\alpha\\ &+2\sum\frac{1}{|G_\alpha|}\int_{\tilde{s}_\alpha}\Big\langle R^N\Big(d\tilde{f}_t(\tilde{e}_i),d\tilde{f}_t\Big(\frac{\partial}{\partial t}\Big)\Big)d\tilde{f}_t(\tilde{e}_i),\big(\tilde{\tilde{D}}_{\tilde{e}_j}d\tilde{f}_t\big)(\tilde{e}_j)\Big\rangle d\tilde{x}_\alpha.\end{aligned}\tag{2.7}$$

Let $t=0$, and by the symmetry of the Riemannian curvature tensor, we derive (2.4). □

Definition 2.3. A smooth map $f:(M,\mathscr{F})\to N$ from a compact V-manifold M into a Riemannian manifold N is biharmonic if and only if

$$\tau_2(\tilde{f})=\triangle\tau(\tilde{f})+R^N(d\tilde{f}(\tilde{e}_i),\tau(\tilde{f}))d\tilde{f}(\tilde{e}_i)=0\tag{2.8}$$

in each $\{\tilde{U},G,\pi\}\in\mathscr{F}$ over the support U on M.

A harmonic map $f:M\to N$ on a V-manifold M is obviously a biharmonic map, but a harmonic map is not necessarily a biharmonic map. However, we obtain the following theorem.

Theorem 2.4. *Suppose that M is a compact V-manifold, and N is a Riemannian manifold of nonpositive curvature. If $f:M\to N$ is a biharmonic map, then f is a harmonic map.*

Proof. In each V-chart $\{\tilde{U},G,\pi\}\in\mathscr{F}$ over the support U on M it is calculated by

$$\begin{aligned}\triangle e_2(\tilde{f})&=\frac{1}{2}\triangle\|\tau(\tilde{f})\|^2=\big\langle\tilde{D}_{\tilde{e}_k}\tau(\tilde{f}),\tilde{D}_{\tilde{e}_k}\tau(\tilde{f})\big\rangle+\big\langle\bar{D}^*\bar{D}\tau(\tilde{f}),\tau(\tilde{f})\big\rangle\\ &=\big\langle\bar{D}_{\tilde{e}_k}\tau(\tilde{f}),\bar{D}_{e_k}\tau(\tilde{f})\big\rangle-\big\langle R^N(d\tilde{f}(\tilde{e}_i),\tau(\tilde{f}))d\tilde{f}(\tilde{e}_i),\tau(\tilde{f})\big\rangle\ge 0,\end{aligned}\tag{2.9}$$

because $\tau_2(\tilde{f})=0$ in each $\tilde{U}$ and the Riemannian curvature of N is nonpositive. By Bochner's technique and the assumption $\partial\tilde{f}_t/\partial t=0$ and $\bar{D}_{\tilde{e}_i}\partial\tilde{f}_t/\partial t=0$ outside a compact subset of $\operatorname{int}(\tilde{U})$, we know $\|\tau(\tilde{f})\|^2=\text{const}$, and then substituting into (2.9) we have $\bar{D}_{\tilde{e}_k}(\tau\tilde{f})=0$, for all $k=1,2,\dots,m$ by [7] which implies $\tau(\tilde{f})=0$ in each $\tilde{U}$, that is, f is harmonic on M. □

Since harmonic maps are automatically biharmonic maps when the Riemannian curvature of N is nonpositive, we will find a non-trivial biharmonic map into a sphere. By the concepts of V-manifolds and the similar techniques as [11], we have the following theorem.

Theorem 2.5. *Let $f:(M,\mathcal{F})\to S^{m+1}$ be nonzero parallel mean curvature isometric embedding, then f is biharmonic if and only if the second fundamental form $B(\tilde{f})$ of $\tilde{f}$ with $\|B(\tilde{f})\|^2=m=\dim(\tilde{U})$ in each $\tilde{U}$ over the support U on M.*

Example 2.6. In S^{m+1}, the compact hypersurface of its Gauss map being isometric embedding is the Clifford surface (see [15]):

$$M_k^m(1)=S^k\left(\sqrt{\frac{1}{2}}\right)\times S^{m-k}\left(\sqrt{\frac{1}{2}}\right),\quad 0\le k\le m. \tag{2.10}$$

Let $f:M_k^m(1)\to S^{m+1}$ be the standard embedding. Set

$$M_k^m(1)'=\frac{S^k(\sqrt{1/2})}{Z_p}\times\frac{S^{m-k}(\sqrt{1/2})}{Z_{p'}}, \tag{2.11}$$

where p, p' are prime numbers (p and p' could be the same or different). Since both the first and the second terms are compact V-manifolds, the product is also a compact V-manifold. Let $f':M_k^m(1)'\to S^{m+1}$ be a map such that $k\ne m/2$, pick $\tilde{U}=\{(x^0,x^1,\dots,x^k)\in S^k\sqrt{1/2}:x^i>0,\ i\text{ is any of }0,1,\dots,k\}\times\{(x^{k+1},\dots,x^{m+1})\in S^{m-k}\sqrt{1/2}:x^j>0,\ j\text{ is any of }k+1,\dots,m+1\}$ (if x^i and x^j vary, $\tilde{U}$ is different), and let $\tilde{f}':\tilde{U}\to S^{m+1}$ (as part of the standard map $f:S^k\sqrt{1/2}\times S^{m-k}\sqrt{1/2}\to S^{m+1}$) in each $\{\tilde{U},G,\pi\}\in\mathcal{F}$. So $\tilde{f}'$ has parallel second fundamental form, and has parallel mean curvature and $B(\tilde{f}')=k+m-k=m,\|\tau(\tilde{f}')\|=|k-(m-k)|=2k-m\ne 0$. That is, $\tilde{f}'$ is biharmonic in $\tilde{U}$ for each $\{\tilde{U},G,\pi\}\in\mathcal{F}$. Then by Theorem 2.5 f is a nontrivial biharmonic map on $(M,\mathcal{F})$.

3. The stability and composition of biharmonic maps on V-manifolds.

Let M be a compact V-manifold, and N a Riemannian manifold. We continue to use the notations as in the previous sections. By applying the Green's divergence theorem on the compact V-manifold M [3], the concepts of V-manifolds, and the similar techniques in [11], we can have the second variations of biharmonic maps as follows.

Theorem 3.1. *If $f:(M\mathcal{F})\to N$ is a biharmonic map, then*

$$\begin{aligned}\frac{1}{2}\frac{d^2}{dt^2}E_2(f_t)\bigg|_{t=0}&\\ =\sum\frac{1}{|G_\alpha|}&\int_{\tilde{s}_\alpha}\|\triangle V+R^N(d\tilde{f}(\tilde{e}_i),V)d\tilde{f}(\tilde{e}_i)\|^2d\tilde{x}_\alpha\\ +\sum\frac{1}{|G_\alpha|}&\int_{\tilde{s}_\alpha}\Big\langle V,(D'_{d\tilde{f}(\tilde{e}_k)}R^N)(d\tilde{f}(\tilde{e}_k),\tau(\tilde{f}))V\\ &+(D'_{\tau(\tilde{f})}R^N)(d\tilde{f}(\tilde{e}_i),V)\,d\tilde{f}(\tilde{e}_i)+R^N(\tau(\tilde{f}),V)\tau(\tilde{f})\\ &+2R^N(d\tilde{f}(\tilde{e}_k),V)\bar{D}_{\tilde{e}_k}\tau(\tilde{f})+2R^N(d\tilde{f}(\tilde{e}_i),\tau(\tilde{f}))\bar{D}_{\tilde{e}_i}V\Big\rangle d\tilde{x}_\alpha.\end{aligned} \tag{3.1}$$

Definition 3.2. Let $f:(M,\mathcal{F})\to N$ be a biharmonic map from a compact V-manifold M into a Riemannian manifold N. If $d^2/dt^2E_2(f_t)|_{t=0}\ge 0$, then f is a *stable* biharmonic map.

If we look at a harmonic map as a biharmonic map, then it must be stable by the definition of bi-energy since

$$\frac{1}{2}\frac{d^2}{dt^2}E_2(f_t)\Big|_{t=0}=\sum\frac{1}{|G_\alpha|}\int_{\tilde{s}_\alpha}\left|\left|\triangle V+R^N(d\tilde{f})((\tilde{e}_i),V)d\tilde{f}(\tilde{e}_i)\right|\right|^2 d\tilde{x}_\alpha\geq 0. \tag{3.2}$$

Theorem 3.3. *Let $f:(M,\mathcal{F})\to N$ be a stable biharmonic map from a compact V-manifold M into a Riemannian manifold N of constant sectional curvature $K>0$ and f satisfies the conservation law, then f must be a harmonic map.*

Proof. Because N has the constant sectional curvature, the term of $D'R^N$ of the second variation formula disappears and

$$\begin{aligned}\frac{1}{2}\frac{d^2}{dt^2}E_2(f_t)\Big|_{t=0}=&\sum\frac{1}{|G_\alpha|}\int_{\tilde{s}_\alpha}\left|\left|\triangle V+R^N(df(e_i),V)df(e_i)\right|\right|^2 d\tilde{x}_\alpha\\&+\sum\frac{1}{|G_\alpha|}\int_{\tilde{s}_\alpha}\Big\langle V,R^N(\tau(\tilde{f}),V)\tau(\tilde{f})+2R^N(d\tilde{f}(\tilde{e}_k),V)\bar{D}_{\tilde{e}_k}\tau(\tilde{f})\\&\qquad+2R^N(d\tilde{f}(\tilde{e}_i),\tau(\tilde{f}))\bar{D}_{\tilde{e}_i}V\Big\rangle d\tilde{x}_\alpha.\end{aligned} \tag{3.3}$$

Take $V=\tau(\tilde{f})$, and notice that f is biharmonic and N has the constant sectional curvature, then by (3.3) we have

$$\begin{aligned}\frac{1}{2}\frac{d^2}{dt^2}E_2(f_t)\Big|_{t=0}&=\sum\frac{4}{|G_\alpha|}\int_{\tilde{s}_\alpha}\Big\langle R^N(d\tilde{f}(\tilde{e}_i),\tau(\tilde{f}))\bar{D}_{\tilde{e}_k}\tau(\tilde{f}),\tau(\tilde{f})\Big\rangle d\tilde{x}_\alpha\\&=4K\sum\frac{1}{|G_\alpha|}\int_{\tilde{s}_\alpha}\Big[\Big\langle d\tilde{f}(\tilde{e}_k),\tilde{\bar{D}}_{\tilde{e}_k}\tau(\tilde{f})\Big\rangle\left|\left|\tau(\tilde{f})\right|\right|^2\\&\qquad-\Big\langle d\tilde{f}(\tilde{e}_k),\tau(\tilde{f})\Big\rangle\Big\langle\tau(\tilde{f}),\bar{D}_{\tilde{e}_k}\tau(\tilde{f})\Big\rangle\Big]d\tilde{x}_\alpha.\end{aligned} \tag{3.4}$$

In each $\tilde{U}_\alpha$, $\tilde{f}$ satisfies the conservation law [2], so

$$\begin{aligned}\Big\langle d\tilde{f}(\tilde{e}_k),\tau(\tilde{f})\Big\rangle&=0,\\\Big\langle d\tilde{f}(\tilde{e}_k),\bar{D}_{\tilde{e}_k}\tau(\tilde{f})\Big\rangle=-\Big\langle\bar{D}_{\tilde{e}_k}d\tilde{f}(\tilde{e}_k),\tau(\tilde{f})\Big\rangle&=-\left|\left|\tau(\tilde{f})\right|\right|^2\end{aligned} \tag{3.5}$$

in each $\tilde{U}$. Substitute (3.5) into (3.4), and f is stable, we have

$$\frac{1}{2}\frac{d^2}{dt^2}E_2(f_t)\Big|_{t=0}=-4K\sum\frac{1}{|G_\alpha|}\int_{\tilde{s}_\alpha}\left|\left|\tau(\tilde{f})\right|\right|^4 d\tilde{x}_\alpha\geq 0. \tag{3.6}$$

Therefore, $\tau(\tilde{f})=0$ in each $\tilde{s}_\alpha$ of $\tilde{U}_\alpha$, that is, f is harmonic on $(M,\mathcal{F})$.

Let $f:(M,\mathcal{F})\to M'$ be a smooth map from a compact V-manifold $(M,\mathcal{F})$ into a Riemannian manifold and M', and $f_1:M'\to M''$ a smooth map from M' into another Riemannian manifold M''. Then the composition $f_1\circ f:M\to M''$ is a smooth map. Let D, D', $\bar{D}$, $\bar{D}'\hat{D}$, $\hat{D}'$, $\hat{D}''$ be the Riemannian connections on TM, TM', $f^{-1}TM$, $f_1^{-1}TM''$, $(f_1\circ f)^{-1}TM''$, $T^*M\otimes f^{-1}TM'$, $T^*M'\otimes f_1^{-1}TM''$, $T^*M\otimes(f_1\circ f)^{-1}TM''$, respectively, and let $R^{M'}(,)$, $R^{f_1^{-1}TM''}$ be the Riemannian curvatures on TM'', $f^{-1}TM''$, respectively. For all $X,Y\in\Gamma(TM)$, we have

$$\bar{D}''_X d(f_1\circ f)Y=\hat{D}'_{df(X)}df_1(Y)+df_1\circ\bar{D}_X df(Y). \tag{3.7}$$

□

Theorem 3.4. *Let $(M,\mathcal{F})$ be a compact V-manifold, and M', M'' Riemannian manifolds. If $f: M \to M'$ is a biharmonic map and $f_1: M' \to M''$ is totally geodesic, then the composition $f_1 \circ f: M \to M''$ is a biharmonic map.*

Proof. Since f_1 is totally geodesic, that is, $\hat{D}' df_1 = 0$, so in each $\tilde{U}$ we have $\tau(f_1 \circ \tilde{f}) = df_1 \circ \tau(\tilde{f})$ and

$$\begin{aligned} \bar{D''}^{*}\bar{D}\tau(f_1\circ\tilde{f}) &= \bar{D''}^{*}\bar{D}''(df_1\circ\tau(\tilde{f})) \\ &= \bar{D}''_{\tilde{e}_k}\bar{D}''_{\tilde{e}_k}(df_1\circ\tau(\tilde{f})) - \bar{D}''_{D_{\tilde{e}_k}\tilde{e}_k}(df_1\circ\tau(\tilde{f})). \end{aligned} \tag{3.8}$$

By (3.7) and notice that f_1 is totally geodesic, then

$$\begin{aligned} \bar{D}''_{\tilde{e}_k}(df_1\circ\tau(\tilde{f})) &= \bar{D}''_{\tilde{e}_k}(df_1\circ\hat{D}_{\tilde{e}_j}d\tilde{f}(\tilde{e}_j)) \\ &= \left(\hat{D}'_{\hat{D}_{\tilde{e}_j}d\tilde{f}(\tilde{e}_k)}df_1\right)(\hat{D}_{\tilde{e}_j}d\tilde{f}(\tilde{e}_j)) + df_1\circ\bar{D}_{\tilde{e}_k}(\hat{D}_{\tilde{e}_j}d\tilde{f}(\tilde{e}_j)) \\ &= df_1\circ\bar{D}_{\tilde{e}_k}\tau(\tilde{f}). \end{aligned} \tag{3.9}$$

So

$$\begin{aligned} &\bar{D}''_{\tilde{e}_k}\bar{D}''_{\tilde{e}_k}(df_1\circ\tau(\tilde{f})) = \bar{D}''_{\tilde{e}_k}(df_1\circ\bar{D}_{\tilde{e}_k}\tau(\tilde{f})) = df_1\circ\bar{D}_{\tilde{e}_k}\bar{D}_{\tilde{e}_k}\tau(\tilde{f}), \\ &\bar{D}''_{D_{\tilde{e}_k}\tilde{e}_k}(df_1\circ\tau(\tilde{f})) = df_1\circ\bar{D}_{D_{\tilde{e}_k}\tilde{e}_k}\tau(\tilde{f}). \end{aligned} \tag{3.10}$$

Substituting (3.10) into (3.8), we get

$$\bar{D''}^{*}\tau(f_1\circ\tilde{f}) = df_1\circ\bar{D}^{*}\bar{D}\tau(\tilde{f}). \tag{3.11}$$

On the other hand,

$$\begin{aligned} R^{M''}&(d(f_1\circ\tilde{f})(\tilde{e}_i),\tau(f_1\circ\tilde{f}))\,d(f_1\circ f)(\tilde{e}_i) \\ &= R^{f_1^{-1}TM''}(d\tilde{f}(\tilde{e}_i),\tau(\tilde{f}))\,df_1(d\tilde{f}(\tilde{e}_i)) \\ &= df_1\circ R^{M'}(d\tilde{f}(\tilde{e}_i),\tau(\tilde{f}))\,d\tilde{f}(\tilde{e}_i). \end{aligned} \tag{3.12}$$

By (3.11) and (3.12), we have

$$\begin{aligned} &\bar{D}^{*}\bar{D}''(f_1\circ\tilde{f}) + R^{M''}(d(f_1\circ\tilde{f})(\tilde{e}_i),\tau(f_1\circ\tilde{f}))\,d(f_1\circ\tilde{f})(\tilde{e}_i) \\ &\qquad = df_1\circ\left[\bar{D}^{*}\bar{D}\tau(\tilde{f}) + R^{M'}(d\tilde{f}(\tilde{e}_i),\tau(\tilde{f}))\,d\tilde{f}(\tilde{e}_i)\right] \end{aligned} \tag{3.13}$$

in each $\tilde{U}$. Hence, if f is biharmonic, then $f_1 \circ f$ is also biharmonic. □

Remark 3.5. Theorem 3.4 generalizes the main theorem in [14] into V-manifolds. The condition of f_1 being totally geodesic cannot be weakened into harmonic or biharmonic.

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References

[1] W. L. Baily Jr., *The decomposition theorem for V-manifolds*, Amer. J. Math. **78** (1956), 862–888. MR 20#6537. Zbl 173.22705.

[2] P. Baird and J. Eells, *A conservation law for harmonic maps*, Geometry Symposium, Utrecht 1980 (Utrecht, 1980), Lecture Notes in Math., vol. 894, Springer, Berlin, 1981, pp. 1–25. MR 83i:58031. Zbl 485.58008.

[3] Y.-J. Chiang, *Harmonic maps of V-manifolds*, Ann. Global Anal. Geom. **8** (1990), no. 3, 315–344. MR 92c:58021. Zbl 679.58014.

[4] ———, *Spectral geometry of V-manifolds and its application to harmonic maps*, Differential Geometry: Partial Differential Equations on Manifolds (Los Angeles, CA, 1990) (Rhode Island), Proc. Sympos. Pure Math., vol. 54, Part 1, Amer. Math. Soc., 1993, pp. 93–99. MR 94c:58040. Zbl 806.58005.

[5] Y.-J. Chiang and A. Ratto, *Harmonic maps on spaces with conical singularities*, Bull. Soc. Math. France **120** (1992), no. 2, 251–262. MR 93h:58040. Zbl 758.53023.

[6] Y-J. Chiang and H. Sun, 2-*harmonic totally real submanifolds in a complex projective space*, Bull. Inst. Math. Acad. Sinica **27** (1999), no. 2, 99–107. MR 2000e:53079. Zbl 960.53036.

[7] J. Eells and L. Lemaire, *A report on harmonic maps*, Bull. London Math. Soc. **10** (1978), no. 1, 1–68. MR 82b:58033. Zbl 401.58003.

[8] ———, *Another report on harmonic maps*, Bull. London Math. Soc. **20** (1988), no. 5, 385–524. MR 89i:58027. Zbl 669.58009.

[9] J. Eells, Jr. and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964), 109–160. MR 29#1603. Zbl 122.40102.

[10] G. Y. Jiang, 2-*harmonic isometric immersions between Riemannian manifolds*, Chinese Ann. Math. Ser. A **7** (1986), no. 2, 130–144 (Chinese). MR 87k:53140. Zbl 596.53046.

[11] ———, 2-*harmonic maps and their first and second variational formulas*, Chinese Ann. Math. Ser. A **7** (1986), no. 4, 389–402 (Chinese). MR 88i:58039. Zbl 0628.58008.

[12] I. Satake, *On a generalization of the notion of manifold*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 359–363. MR 18,144a. Zbl 074.18103.

[13] ———, *The Gauss-Bonnet theorem for V-manifolds*, J. Math. Soc. Japan **9** (1957), 464–492. MR 20#2022. Zbl 080.37403.

[14] H. Sun, *A theorem on* 2-*harmonic mappings*, J. Math. (Wuhan) **12** (1992), no. 1, 103–106 (Chinese). MR 94c:58045. Zbl 0766.53036.

[15] Y. L. Xin and X. P. Chen, *The hypersurfaces in the Euclidean sphere with relative affine Gauss maps*, Acta Math. Sinica **28** (1985), no. 1, 131–139 (Chinese). MR 87b:53088. Zbl 0567.53041.

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