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# FIXED POINT THEOREMS FOR GENERALIZED LIPSCHITZIAN SEMIGROUPS

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ABSTRACT. Let $K$ be a nonempty subset of a $p$-uniformly convex Banach space $E$, $G$ a left reversible semitopological semigroup, and $\mathcal{S} = \{T_t : t \in G\}$ a generalized Lipschitzian semigroup of $K$ into itself, that is, for $s \in G$, $\|T_s x - T_s y\| \le a_s\|x-y\| + b_s(\|x - T_s x\| + \|y - T_s y\|) + c_s(\|x - T_s y\| + \|y - T_s x\|)$, for $x, y \in K$ where $a_s, b_s, c_s > 0$ such that there exists a $t_1 \in G$ such that $b_s + c_s < 1$ for all $s \succeq t_1$. It is proved that if there exists a closed subset $C$ of $K$ such that $\bigcap_s \overline{\text{co}}\{T_t x : t \succeq s\} \subset C$ for all $x \in K$, then $\mathcal{S}$ with $[(\alpha+\beta)^p(\alpha^p \cdot 2^{p-1} - 1)/(c_p - 2^{p-1}\beta^p) \cdot N^p]^{1/p} < 1$ has a common fixed point, where $\alpha = \limsup_s (a_s + b_s + c_s)/(1 - b_s - c_s)$ and $\beta = \limsup_s (2b_s + 2c_s)/(1 - b_s - c_s)$.

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**1. Introduction.** Let $K$ be a nonempty subset of a Banach space $E$ and $T$ a mapping of $K$ into itself. The mapping $T$ is said to be Lipschitzian mapping if for each $n \ge 1$, there exists a positive real number $k_n$ such that

$$\|T^n x - T^n y\| \le k_n \|x - y\| \tag{1.1}$$

for all $x$, $y$ in $K$. A Lipschitzian mapping is said to be nonexpansive if $k_n = 1$ for all $n \ge 1$, uniformly $k$-Lipschitzian if $k_n = k$ for all $n \ge 1$, and asymptotically nonexpansive if $\lim_n k_n = 1$, respectively. These mappings were first studied by Goebel and Kirk [7] and Goebel, Kirk, and Thele [9]. Lifšic [13] proved that in a Hilbert space a uniformly $k$-Lipschitzian mapping with $k < \sqrt{2}$ has a fixed point. Downing and Ray [4] and Ishihara and Takahashi [12] proved that in a Hilbert space a uniformly $k$-Lipschitzian semigroup with $k < \sqrt{2}$ has a common fixed point. Casini and Maluta [3] and Ishihara and Takahashi [11] proved that a uniformly $k$-Lipschitzian semigroup in a Banach space $E$ has a common fixed point if $k < \sqrt{N(E)}$, where $N(E)$ is the constant of uniformly normal structure.

In these results, the domains of semigroups were assumed to be closed and convex. Ishihara [10] gave fixed point theorems for Lipschitzian semigroups in both Banach and Hilbert spaces in which closedness and convexity of domain were not needed.

Now we consider the following class of mappings, which we call generalized Lipschitzian mapping, whose $n$th iterate $T^n$ satisfying the following condition:

$$\begin{aligned} \|T^n x - T^n y\| \le a_n\|x-y\| + b_n(\|x - T^n x\| + \|y - T^n y\|) \\ + c_n(\|x - T^n y\| + \|y - T^n x\|), \end{aligned} \tag{1.2}$$

for each $x, y \in K$ and $n \ge 1$, where $a_n$, $b_n$, $c_n$ are the nonnegative constants such that there exists an integer $n_0$ such that $b_n + c_n < 1$ for all $n \ge n_0$.

This class of generalized Lipschitzian mappings is more general than the classes of nonexpansive, asymptotically nonexpansive, Lipschitzian and uniformly $k$-Lipschitzian mappings. The above facts can be seen by taking $b_n = c_n = 0$.

In this paper, we prove a fixed point theorem for generalized Lipschitzian semigroups in a $p$-uniformly convex Banach space. Next we give its corollaries in a Hilbert space, in $L^p$ spaces, in Hardy space $H^p$ and in Sobolev spaces $H^{k,p}$, for $1 < p < \infty$ and $k \geq 0$. Our results improve and extend results from [10, 11, 12].

**2. Preliminaries.** Let $G$ be a semitopological semigroup, that is, $G$ is a semigroup with a Hausdorff topology such that for each $a \in G$ the mapping $s \to a \cdot s$ and $s \to s \cdot a$ from $G$ to $G$ are continuous. A semitopological semigroup $G$ is left reversible if any two closed right ideals of $G$ have nonempty intersection. In this case, $(G, \preceq)$ is a directed system when the binary relation "$\preceq$" on $G$ is defined by $a \preceq b$ if and only if $\{a\} \cup \overline{aG} \supseteq \{b\} \cup \overline{bG}$. Examples of left reversible semigroups include commutative and all left amenable semigroups.

Let $K$ be a mapping subset of a Banach space $E$. Then a family $\mathcal{S} = \{T_t : t \in G\}$ of mappings from $K$ into itself is said to be a generalized Lipschitzian semigroup on $K$ if $\mathcal{S}$ satisfies the following:

(i) $T_{ts}(x) = T_t T_s(x)$ for $t,\ s \in G$ and $x \in K$;

(ii) the mapping $(s, x) \to T_s(x)$ from $G \times K$ into $K$ is continuous when $G \times K$ has the product topology;

(iii) for each $s \in G$

$$||T_s x - T_s y|| \leq a_s \|x - y\| + b_s(||x - T_s x|| + ||y - T_s y||) + c_s(||x - T_s y|| + ||y - T_s x||), \tag{2.1}$$

for $x, y \in K$ where $a_s, b_s, c_s > 0$ such that there exists a $t_1 \in G$ such that $b_s + c_s < 1$ for all $s \succeq t_1$.

Let $\{B_\alpha : \alpha \in \wedge\}$ be a decreasing net of bounded subsets of a Banach space $E$. For a nonempty subset $K$ of $E$, define

$$\begin{aligned} r(\{B_\alpha\}, x) &= \inf_\alpha \sup\{\|x - y\| : y \in B_\alpha\}; \\ r(\{B_\alpha\}, K) &= \inf\{r(\{B_\alpha\}, x) : x \in K\}; \\ A(\{B_\alpha\}, K) &= \{x \in K : r(\{B_\alpha\}, x) = r(\{B_\alpha\}, K)\}. \end{aligned} \tag{2.2}$$

We know that $r(\{B_\alpha\}, \cdot)$ is a continuous convex function on $E$ which satisfies the following:

$$|r(\{B_\alpha\}, x) - r(\{B_\alpha\}, y)| \leq \|x - y\| \leq r(\{B_\alpha\}, x) + r(\{B_\alpha\}, y) \tag{2.3}$$

for each $x, y \in E$. It is easy to see that if $E$ is reflexive and $K$ is closed convex, then $A(\{B_\alpha\}, K)$ is nonempty, and moreover, if $E$ is uniformly convex, then it consists of a single point (cf. [14]).

Let $p > 1$, and denote by $\lambda$ the number in $[0, 1]$ and by $W_p(\lambda)$ the function $\lambda \cdot (1 - \lambda)^p + \lambda^p \cdot (1 - \lambda)$.

The functional $\|\cdot\|^p$ is said to be uniformly convex (cf. Zălinescu [25]) on the Banach space $E$ if there exists a positive constant $c_p$ such that for all $\lambda \in [0,1]$ and $x, y \in E$ the following inequality holds:

$$\|\lambda x + (1-\lambda)y\|^p \le \lambda\|x\|^p + (1-\lambda)\|y\|^p - W_p(\lambda)\cdot c_p \cdot \|x-y\|^p. \tag{2.4}$$

Xu [24] proved that the functional $\|\cdot\|^p$ is uniformly convex on the whole Banach space $E$ if and only if $E$ is $p$-uniformly convex, that is, there exists a constant $c > 0$ such that the moduli of convexity (see [8]) $\delta_E(\varepsilon) \ge c\cdot\varepsilon^p$ for all $0 \le \varepsilon \le 2$.

The normal structure coefficient $N(E)$ of $E$ (cf. [2]) is defined by

$$N(E) = \inf\left\{\frac{\operatorname{diam} K}{r_K(K)} : K \text{ is a bounded convex subset of } E \text{ consisting of more than one point}\right\}, \tag{2.5}$$

where $\operatorname{diam} K = \sup\{\|x-y\| : x, y \in K\}$ is the diameter of $K$ and $r_K(K) = \inf_{x\in K}\{\sup_{y\in K}\|x-y\|\}$ is the Chebyshev radius of $K$ relative to itself. The space $E$ is said to have uniformly normal structure if $N(E) > 1$. It is known that a uniformly convex Banach space has uniformly normal structure and for a Hilbert space $H$, $N(H) = \sqrt{2}$. Recently, Pichugov [18] (cf. Prus [20]) calculated that

$$N(L^p) = \min\{2^{1/p}, 2^{(p-1)/p}\}, \quad 1 < p < \infty. \tag{2.6}$$

Some estimates for normal structure coefficients in other Banach spaces may be found in [21].

For a subset $K$, we denote by $\overline{\text{co}}K$ the closure of the convexity hull of $K$.

## 3. Main results.

Now we are in position to give our result.

**Theorem 3.1.** *Let $p > 1$ and let $E$ be a $p$-uniformly convex Banach space, $K$ a nonempty subset of $E$, $G$ a left reversible semitopological semigroup, and $\mathcal{S} = \{T_t : t \in G\}$ a generalized Lipschitzian semigroup on $K$ with*

$$\left[\frac{(\alpha+\beta)^p(\alpha^p\cdot 2^{p-1}-1)}{(c_p - 2^{p-1}\beta^p)\cdot N^p}\right]^{1/p} < 1, \tag{3.1}$$

*where*

$$\alpha = \limsup_s \frac{a_s+b_s+c_s}{1-b_s-c_s}, \qquad \beta = \limsup_s \frac{2b_s+2c_s}{1-b_s-c_s}. \tag{3.2}$$

*Suppose that $\{T_t y : t \in G\}$ is bounded for some $y \in K$ and there exists a closed subset $C$ of $K$ such that $\bigcap_s \overline{\text{co}}\{T_t x : t \succeq s\} \subseteq C$ for all $x \in K$. Then there exists a $z \in C$ such that $T_s z = z$ for all $s \in G$.*

**Proof.** Let $B_s(x) = \overline{\text{co}}\{T_t x : t \succeq s\}$ and let $B(x) = \bigcap_s B_s(x)$ for $s \in G$ and $x \in K$. Define $\{x_n : n \ge 0\}$ by induction as follows:

$$x_0 = y, \qquad x_n = A(\{B_s(x_{n-1})\}, B(x_{n-1})), \quad \text{for } n \ge 1. \tag{3.3}$$

Since $B(x) \subseteq C \subseteq K$ for all $x \in K$, $\{x_n\}$ is well defined. Let

$$r_m = r(\{B_s(x_m)\}, B(x_m)),$$
$$D_m = r(\{B_s(x_m)\}, B(x_{m-1})), \quad m \geq 1. \tag{3.4}$$

Now, for each $s,\ t \in G$ and $x,\ y \in K$, we have

$$\begin{aligned}||T_sT_tx - T_sy|| \leq a_s||T_tx - y|| + b_s(||T_tx - T_sT_tx|| + ||y - T_sy||)\\ + c_s(||y - T_sT_tx|| + ||T_tx - T_sy||),\end{aligned} \tag{3.5}$$

and so

$$||T_sT_tx - T_sy|| \leq \frac{a_s + b_s + c_s}{1 - b_s - c_s} \cdot ||T_tx - y|| + \frac{2b_s + 2c_s}{1 - b_s - c_s} \cdot ||y - T_sy||. \tag{3.6}$$

Then from $x_m \in B(x_{m-1}) = \bigcap_t B_t(x_{m-1})$ and a result of Ishihara and Takahashi [11], we have

$$r_m = r(\{B_s(x_m)\}, B(x_m)) \leq \frac{1}{N} \cdot \inf_s \operatorname{diam}(B_s(x_m)) \tag{3.7}$$

and by using (3.6), we have

$$\begin{aligned}\inf_s \operatorname{diam}(B_s(x_m)) &= \inf_s \sup\{||T_ax_m - T_bx_m|| : a,\ b \succeq s\}\\ &\leq \limsup_t (\limsup_s ||T_sx_m - T_tx_m||)\\ &\leq \limsup_t (\limsup_s ||T_tT_sx_m - T_tx_m||)\\ &\leq \limsup_t \left[\limsup_s \left\{\left(\frac{a_t + b_t + c_t}{1 - b_t - c_t}\right) \cdot ||T_sx_m - x_m|| \right.\right.\\ &\qquad \left.\left. + \left(\frac{2b_t + 2c_t}{1 - b_t - c_t}\right) \cdot ||x_m - T_tx_m||\right\}\right]\\ &\leq (\alpha + \beta) \cdot D_m,\end{aligned} \tag{3.8}$$

and hence

$$r_m \leq \frac{\alpha + \beta}{N} \cdot D_m, \tag{3.9}$$

where $N$ is the normal structure coefficient of $E$. Again from (2.4) and (3.6) we have

$$\begin{aligned}&||\lambda x_{m+1} + (1 - \lambda)T_tx_{m+1} - T_sx_m||^p + c_p \cdot W_p(\lambda) \cdot ||x_{m+1} - T_tx_{m+1}||^p\\ &\quad \leq \lambda||x_{m+1} - T_sx_m||^p + (1 - \lambda) \cdot ||T_tx_{m+1} - T_sx_m||^p\\ &\quad \leq \lambda||x_{m+1} - T_sx_m||^p + (1 - \lambda) \cdot ||T_tx_{m+1} - T_tT_sx_m||^p\\ &\quad \leq \lambda||x_{m+1} - T_sx_m||^p + (1 - \lambda) \cdot \left[\frac{a_t + b_t + c_t}{1 - b_t - c_t} \cdot ||T_sx_m - x_{m+1}||\right.\\ &\qquad \left. + \frac{2b_t + 2c_t}{1 - b_t - c_t} \cdot ||T_tx_{m+1} - x_{m+1}||\right]^p.\end{aligned} \tag{3.10}$$

Taking the $\limsup_s$, we have

$$r_m^p + c_p \cdot W_p(\lambda) \cdot ||x_{m+1} - T_t x_{m+1}||^p \\ \le \lambda r_m^p + (1-\lambda)\left[\frac{a_t + b_t + c_t}{1 - b_t - c_t} \cdot r_m + \frac{2b_t + 2c_t}{1 - b_t - c_t} \cdot ||T_t x_{m+1} - x_{m+1}||\right]^p. \tag{3.11}$$

It then follows that

$$r_m^p + c_p \cdot W_p(\lambda) \cdot D_{m+1}^p \le \lambda r_m^p + (1-\lambda) \cdot 2^{p-1}\left[\alpha^p r_m^p + \beta^p \cdot D_{m+1}^p\right], \tag{3.12}$$

and so

$$D_{m+1}^p \le \left[\frac{(1-\lambda)\cdot(2^{p-1}\cdot\alpha^p - 1)}{c_p \cdot W_p(\lambda) - (1-\lambda)\cdot 2^{p-1}\cdot\beta^p}\right]\cdot r_m^p \\ \le \left[\frac{(1-\lambda)\cdot(2^{p-1}\cdot\alpha^p - 1)}{c_p \cdot W_p(\lambda) - (1-\lambda)\cdot 2^{p-1}\cdot\beta^p}\right]\cdot \frac{(\alpha+\beta)^p}{N^p}\cdot D_m^p. \tag{3.13}$$

Letting $\lambda \to 1$, we conclude that

$$D_{m+1} \le \left[\frac{(\alpha+\beta)^p(2^{p-1}\cdot\alpha^p - 1)}{(c_p - 2^{p-1}\cdot\beta^p)\cdot N^p}\right]^{1/p}\cdot D_m = A \cdot D_m, \quad m \ge 1, \tag{3.14}$$

where

$$A = \left[\frac{(\alpha+\beta)^p(2^{p-1}\cdot\alpha^p - 1)}{(c_p - 2^{p-1}\cdot\beta^p)\cdot N^p}\right]^{1/p} < 1 \tag{3.15}$$

by the assumption of the theorem. Since

$$\begin{aligned} ||x_{m+1} - x_m|| &\le r(\{B_s(x_m)\}, x_{m+1}) + r(\{B_s(x_m)\}, x_m) \\ &\le r_m + D_m \\ &\le 2D_m \\ &\vdots \\ &\le 2\cdot A^{m-1} D_1 \longrightarrow 0 \quad \text{as } m \longrightarrow \infty, \end{aligned} \tag{3.16}$$

it follows that $\{z_m\}$ is a Cauchy sequence. Let $z = \lim_{m\to\infty} x_m$. Then we have

$$\begin{aligned} ||z - T_s z|| &\le ||z - x_m|| + ||x_m - T_s x_m|| + ||T_s x_m - T_s z|| \\ &\le ||z - x_m|| + ||x_m - T_s x_m|| + \frac{a_s + b_s + c_s}{1 - b_s - c_s}||z - x_m|| \\ &\quad + \frac{2b_s + 2c_s}{1 - b_s - c_s}||x_m - T_s x_m|| \\ &\le \frac{1 + a_s}{1 - b_s - c_s}\cdot ||z - x_m|| + \frac{1 + b_s + c_s}{1 - b_s - c_s}\cdot ||x_m - T_s x_m||. \end{aligned} \tag{3.17}$$

Taking the limit as $m \to \infty$ on each side, we have

$$||z - T_s z|| \le \lim_{m\to\infty} \left[ \frac{1+a_s}{1-b_s-c_s} \cdot ||z - x_m|| + \frac{1+b_s+c_s}{1-b_s-c_s} \cdot D_m \right] = 0 \tag{3.18}$$

for all $s \in G$. Hence we have $T_s z = z$ for all $s \in G$. This completes the proof. □

**Remark 3.2.** Theorem 3.1 is also true for Lipschitzian semigroup $\mathcal{S} = \{T_t : t \in G\}$ on $K$ with

$$\limsup_s k_s < \left[ \frac{1}{2}\left(1 + \sqrt{1 + 4 \cdot c_p \cdot N^p}\right) \right]^{1/p}. \tag{3.19}$$

As a direct consequence of Theorem 3.1, we have the following result.

**Corollary 3.3.** *Let $p > 1$ and let $E$ be a $p$-uniformly convex Banach space, $K$ a nonempty subset of $E$, and $T$ a mapping from $K$ into itself such that*

$$\begin{aligned} ||T^n x - T^n y|| &\le a_n \|x - y\| + b_n(||x - T^n x|| + ||y - T^n y||) \\ &\quad + c_n(||x - T^n y|| + ||y - T^n x||), \end{aligned} \tag{3.20}$$

*for each $x, y \in K$ and $n \ge 1$, where $a_n$, $b_n$, $c_n$ are the nonnegative constants such that there exists an integer $n_0$ such that $b_n + c_n < 1$ for all $n \ge n_0$. Suppose that $\{T^n y : n \ge 1\}$ is bounded for some $y \in K$ and there exists a closed subset $C$ of $K$ such that $\bigcap_n \overline{\text{co}}\{T^n x : k \ge n\} \subseteq C$ for all $x \in K$. If*

$$\left[ \frac{(\alpha+\beta)^p(\alpha^p \cdot 2^{p-1} - 1)}{(c_p - 2^{p-1}\beta^p) \cdot N^p} \right]^{1/p} < 1, \tag{3.21}$$

*where*

$$\alpha = \limsup_n \frac{a_n + b_n + c_n}{1 - b_n - c_n}, \qquad \beta = \limsup_n \frac{2b_n + 2c_n}{1 - b_n - c_n}, \tag{3.22}$$

*then there exists a $z \in C$ such that $Tz = z$.*

**4. Some applications.** In a Hilbert space $H$, the following equality holds:

$$||\lambda x + (1-\lambda) y||^2 = \lambda \|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2 \tag{4.1}$$

for all $x$, $y$ in $H$ and $\lambda \in [0,1]$.

By Theorem 3.1 and (4.1), we immediately obtain the following.

**Theorem 4.1.** *Let $K$ be a nonempty subset of a Hilbert space $H$, $G$ a left reversible semitopological semigroup, and $\mathcal{S} = \{T_t : t \in G\}$ a generalized Lipschitzian semigroup on $K$ with*

$$\left[ \frac{(\alpha+\beta)^2(2\alpha^2 - 1)}{2(1 - 2\beta^2)} \right]^{1/2} < 1, \tag{4.2}$$

*where $\alpha$, $\beta$ are as in Theorem 3.1. Suppose that $\{T_t y : t \in G\}$ is bounded for some $y \in K$ and there exists a closed subset $C$ of $K$ such that $\bigcap_s \overline{\text{co}}\{T_t x : t \ge s\} \subseteq C$ for all $x \in K$. Then there exists $z \in C$ such that $T_s z = z$ for all $s \in G$.*

The following result follows easily from Theorem 4.1.

**Corollary 4.2** (see [10, Theorem 1]). *Let $K$ be a nonempty subset of a Hilbert space $H$, $G$ a left reversible semitopological semigroup, and $\mathcal{S} = \{T_t : t \in G\}$ a Lipschitzian semigroup on $K$ with $\limsup_s k_s < \sqrt{2}$. Suppose that $\{T_t y : t \in G\}$ is bounded for some $y \in K$ and there exists a closed subset $C$ of $K$ such that $\bigcap_s \overline{\text{co}}\{T_t x : t \succeq s\} \subseteq C$ for all $x \in K$. Then there exists $z \in C$ such that $T_s z = z$ for all $s \in G$.*

If $1 < p \leq 2$, then we have for all $x, y$ in $L^p$ and $\lambda \in [0,1]$,

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)(p-1)\|x-y\|^2. \tag{4.3}$$

(The inequality (4.3) is contained in [16, 23].)

Assume that $2 < p < \infty$ and $t_p$ is the unique zero of the function $g(x) = -x^{p-1} + (p-1)x + p - 2$ in the interval $(1, \infty)$. Let

$$c_p = (p-1)(1+t_p)^{2-p} = \frac{1+t_p^{p-1}}{(1+t_p)^{p-1}}. \tag{4.4}$$

Then we have the following inequality:

$$\|\lambda x + (1-\lambda)y\|^p \leq \lambda\|x\|^p + (1-\lambda)\|y\|^p - W_p(\lambda)\cdot c_p \cdot \|x-y\|^p \tag{4.5}$$

for all $x$, $y$ in $L^p$ and $\lambda \in [0,1]$. (Inequality (4.5) is essentially due to Lim [15].)

By inequality (4.3) and (4.5), we immediately obtain from Theorem 3.1 the following result.

**Theorem 4.3.** *Let $K$ be a nonempty subset of $L^p$, $1 < p < \infty$, $G$ a left reversible semitopological semigroup, and $\mathcal{S} = \{T_t : t \in G\}$ a generalized Lipschitzian semigroup on $K$ with*

$$\begin{aligned} &\left[\frac{(\alpha+\beta)^2(2\alpha^2-1)}{2^{(p-1)/p}(p-1-2\beta^2)}\right]^{1/2} < 1 \quad \text{for } 1 < p \leq 2, \\ &\left[\frac{(\alpha+\beta)^p\cdot(2^{p-1}\alpha^p-1)}{(c_p-2^{p-1}\beta^p)\cdot 2}\right]^{1/p} < 1 \quad \text{for } 2 < p < \infty, \end{aligned} \tag{4.6}$$

*where $\alpha$, $\beta$ are as in Theorem 2.4. Suppose that $\{T_t y : t \in G\}$ is bounded for some $y \in K$ and there exists a closed subset $C$ of $K$ such that $\bigcap_s \overline{\text{co}}\{T_t x : t \succeq s\} \subseteq C$ for all $x \in K$. Then there exists $z \in C$ such that $T_s z = z$ for all $s \in G$.*

**Remark 4.4.** Theorem 4.1 is also true for Lipschitzian semigroup $\mathcal{S} = \{T_t : t \in G\}$ on $K$ with

$$\begin{aligned} &\limsup_s k_s < \left[\frac{1}{2}\left(1+\sqrt{1+4\cdot(p-1)\cdot 2^{(p-1)/p}}\right)\right]^{1/p} \quad \text{for } 1 < p \leq 2, \\ &\limsup_s k_s < \left[\frac{1}{2}\left(1+\sqrt{1+8\cdot c_p}\right)\right]^{1/p} \quad \text{for } 2 < p < \infty. \end{aligned} \tag{4.7}$$

Let $H^p$, $1 < p < \infty$, denote the Hardy space [6] of all functions $x$ analytic in unit disc $|z| < 1$ of the complex plane and such that

$$\|x\| = \lim_{r \to 1^-} \left( \frac{1}{2\pi} \int_0^{2\pi} \left| x(re^{i\theta}) \right|^p d\theta \right)^{1/p} < \infty. \tag{4.8}$$

Now, let $\Omega$ be an open subset of $\mathbb{R}^n$. Denote by $H^{k,p}(\Omega)$, $k \geq 0$, $1 < p < \infty$, the Sobolev space [1, page 149] of distributions $x$ such that $D^\alpha x \in L^p(\Omega)$ for all $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq k$ equipped with the norm

$$\|x\| = \left( \sum_{|\alpha| \leq k} \int_\Omega \left| D^\alpha x(\omega) \right|^p d\omega \right)^{1/p}. \tag{4.9}$$

Let $(\Omega_\alpha, \sum_\alpha, \mu_\alpha)$, $\alpha \in \Lambda$, be a sequence of positive measure spaces, where the index set $\Lambda$ is finite or countable. Given a sequence of linear subspaces $X_\alpha$ in $L^p(\Omega_\alpha, \sum_\alpha, \mu_\alpha)$, we denote by $L_{q,p}$, $1 < p < \infty$ and $q = \max\{2, p\}$ (see [17]), the linear space of all sequences $x = \{x_\alpha \in X_\alpha : \alpha \in \Lambda\}$ equipped with the norm

$$\|x\| = \left( \sum_{\alpha \in \Lambda} \left( \|x_\alpha\|_{p,\alpha} \right)^q \right)^{1/q}, \tag{4.10}$$

where $\|\cdot\|_{p,\alpha}$ denotes the norm in $L^p(\Omega_\alpha, \sum_\alpha, \mu_\alpha)$.

Finally, let $L_p = L^p(S_1, \sum_1, \mu_1)$ and $L_q = L^q(S_2, \sum_2, \mu_2)$, where $1 < p < \infty$, $q = \max\{2, p\}$ and $(S_i, \sum_i, \mu_i)$ are positive measure spaces. Denote by $L_q(L_p)$ the Banach spaces [5, Chapter III, Section 2, Definition 10] of all measurable $L_p$-value function $x$ on $S_2$ such that

$$\|x\| = \left( \int_{S_2} \left( \|x(s)\|_p \right)^q \mu_2(ds) \right)^{1/q}. \tag{4.11}$$

These spaces are $q$-uniformly convex with $q = \max\{2, p\}$ (see [19, 22]), and the norm in these spaces satisfies

$$\|\lambda x + (1-\lambda) y\|^q \leq \lambda \|x\|^q + (1-\lambda) \|y\|^q - d \cdot W_q(\lambda) \cdot \|x - y\|^q \tag{4.12}$$

with a constant

$$d = d_p = \begin{cases} \dfrac{p-1}{8} & \text{for } 1 < p \leq 2, \\ \dfrac{1}{p \cdot 2^p} & \text{for } 2 < p < \infty. \end{cases} \tag{4.13}$$

Now from Theorem 3.1, we have the following result.

**Theorem 4.5.** *Let $K$ be a nonempty subset of the space $E$, where $E = H^p$, or $E = H^{k,p}(\Omega)$, or $E = L_{q,p}$, or $E = L_q(L_p)$, and $1 < p < \infty$, $q = \max\{2, p\}$, $k \geq 0$. Let $G$ be a left reversible semitopological semigroup and $\mathcal{S} = \{T_t : t \in G\}$ a generalized Lipschitzian semigroup on $K$ with*

$$\left[ \frac{(\alpha + \beta)^q (\alpha^q \cdot 2^{q-1} - 1)}{(d - 2^{q-1} \beta^q) \cdot N^q} \right]^{1/q} < 1, \tag{4.14}$$

*where* $\alpha$, $\beta$ *are as in Theorem 3.1. Suppose that* $\{T_t y : t \in G\}$ *is bounded for some* $y \in K$ *and there exists a closed subset* $C$ *of* $K$ *such that* $\bigcap_s \overline{\mathrm{co}}\{T_t x : t \geq s\} \subseteq C$ *for all* $x \in K$*. Then there exists* $z \in C$ *such that* $T_s z = z$ *for all* $s \in G$*.*

**Remark 4.6.** Theorem 4.5 is also true for Lipschitzian semigroup $\mathcal{S} = \{T_t y : t \in G\}$ on $K$ with

$$\limsup_s k_s < \left[\frac{1}{2}\left(1 + \sqrt{1 + 4 \cdot d \cdot N^q}\right)\right]^{1/q}. \tag{4.15}$$

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