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DUAL PAIRS OF SEQUENCE SPACES

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ABSTRACT. The paper aims to develop for sequence spaces E a general concept for reconciling certain results, for example inclusion theorems, concerning generalizations of the Köthe-Toeplitz duals $E^\times$ ($\times \in \{\alpha,\beta\}$) combined with dualities (E,G), $G \subset E^\times$, and the SAK-property (weak sectional convergence). Taking $E^\beta := \{(y_k) \in \omega := \mathbb{K}^\mathbb{N} \mid (y_k x_k) \in cs\} =: E^{cs}$, where cs denotes the set of all summable sequences, as a starting point, then we get a general substitute of E^{cs} by replacing cs by any locally convex sequence space S with sum $s \in S'$ (in particular, a sum space) as defined by Ruckle (1970). This idea provides a dual pair (E,E^S) of sequence spaces and gives rise for a generalization of the solid topology and for the investigation of the continuity of quasi-matrix maps relative to topologies of the duality (E,E^β). That research is the basis for general versions of three types of inclusion theorems: two of them are originally due to Bennett and Kalton (1973) and generalized by the authors (see Boos and Leiger (1993 and 1997)), and the third was done by Große-Erdmann (1992). Finally, the generalizations, carried out in this paper, are justified by four applications with results around different kinds of Köthe-Toeplitz duals and related section properties.

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1. Introduction. In summability as well as in investigations of topological sequence spaces E the duality (E,E^β), where E^β denotes the β-dual of E, plays an essential role. For example, if an FK-space E has the SAK-property (weak sectional convergence), then the topological dual E' can be identified with E^β. Further and more deep-seated connections between topological properties of the dual pair (E,E^β) and the SAK-property, the continuity of matrix maps on E and the structure of domains of matrix methods have been presented, for example, in well-known inclusion theorems by Bennett and Kalton [3, Theorems 4 and 5] (see also [5, 6] for generalizations).

The SAK-property has been generalized and modified by several authors in different directions whereby several generalizations and modifications of the notion of the β-dual has been treated: Buntinas [8] and Meyers [15] as well as further authors have investigated the STK-property (weak T-sectional convergence) in K-spaces E and the corresponding $\beta(T)$-duals, where T is an Sp_1-matrix; Fleming and DeFranza [10, 11] have dealt with the $USTK$-property (unconditionally weak T-sectional convergence) and the corresponding $\alpha(T)$-dual in case of an Sp_1^*-matrix T. That complex of problems is also connected with the $USAK$-property of sequence spaces (cf. Sember [19], Sember and Raphael [18] as well as Swartz [20], Swartz and Stuart [21]), in particular properties of the duality (E,E^α) where E^α denotes the α-dual of E; moreover, Buntinas and Tanović-Miller [9] investigated the strong SAK-property of FK-spaces.

In the present paper, we define and investigate—on the base of the general notion of a sum introduced by Ruckle [17]—dual pairs (E,E^S) where E is a sequence space, S is a K-space on which a sum is defined in the sense of Ruckle, and E^S is the linear space of all corresponding factor sequences. In this connection we introduce and study in Sections 3 and 5 the SK-property which corresponds with the SAK-property and the so-called quasi-matrix maps $\mathfrak{A}$, respectively. In particular, we describe the continuity of $\mathfrak{A}$ and a natural topological structure of the domain $F_{\mathfrak{A}}$ of $\mathfrak{A}$ where F is a K-space. By means of those results, in Section 6 we formulate and prove in that general situation the mentioned inclusion theorems as well as a further theorem of Bennett-Kalton type due to Große-Erdmann [12]. The fact that all mentioned modifications of the SAK-property and of the β-dual are special cases of the SK-property and the factor sequence space E^S, respectively, enables us to deduce in Section 7 from the general inclusion theorems, proved in this paper, those in the listed special cases.

2. Notation and preliminaries. The terminology from the theory of locally convex spaces and summability is standard, we refer to Wilansky [23, 24].

For a given dual pair (E,F) of linear spaces E and F over $\mathbb{K}$ ($\mathbb{K} := \mathbb{R}$ or $\mathbb{C}$) we denote by $\sigma(E,F)$, $\tau(E,F)$, and $\beta(E,F)$ the weak topology, the Mackey topology and the strong topology, respectively. If (E,τ_E) is a given locally convex space, then E^* and E' denotes respectively, the algebraic dual of E and the topological dual of (E,τ_E).

A sequence space is a (linear) subspace of the space ω of all complex (or real) sequences $x = (x_k)$. The sequence space φ is defined to be the set of all finitely nonzero sequences. Obviously, $\varphi = \operatorname{span}\{e^k \mid k \in \mathbb{N}\}$, where $e^k := (0,\dots,0,1,0\dots)$ with "1" in the kth position, and φ contains obviously for each $x \in \omega$ its sections $x^{[n]} := \sum_{k=1}^n x_k e^k$ ($n \in \mathbb{N}$).

If a sequence space E carries a locally convex topology such that the coordinate functionals π_n ($n \in \mathbb{N}$) defined by $\pi_n(x) = x_n$ are continuous, then E is called a K-space. For every K-space E the space φ is a $\sigma(E',E)$-dense subspace of E' where φ is identified with $\operatorname{span}\{\pi_n \mid n \in \mathbb{N}\}$. A K-space which is a Fréchet (Banach) space is called an *FK-(BK-)*space. The sequence spaces

$$\begin{aligned}
m &:= \Big\{x \in \omega \mid \|x\|_\infty := \sup_k |x_k| < \infty\Big\},\\
c &:= \Big\{x \in \omega \mid (x_k) \text{ converges, that is } \lim x := \lim_k x_k \text{ exists}\Big\},\\
c_0 &:= \{x \in c \mid \lim x = 0\},\\
cs &:= \Big\{x \in \omega \,\Big|\, \sum_k x_k \text{ converges}\Big\},\\
\ell &:= \Big\{x \in \omega \,\Big|\, \sum_k |x_k| < \infty\Big\},\\
bv &:= \Big\{x \in \omega \,\Big|\, \sum_k |x_k - x_{k+1}| < \infty\Big\},
\end{aligned} \tag{2.1}$$

(together with their natural norm) are important as well as well-known examples of

BK-spaces. Furthermore, ω is an *FK*-space where its (unique) *FK*-topology is generated by the family of semi-norms r_k, $r_k(x) := |x_k|$ $(x \in \omega,\ k \in \mathbb{N})$.

For sequence spaces E and F we use the notation

$$\begin{aligned} E \cdot F &:= \{yx := (y_k x_k) \mid y \in E,\ x \in F\}, \\ E^F &:= \{y \in \omega \mid \forall x \in E : yx \in F\}. \end{aligned} \tag{2.2}$$

In this way, the well-known α-dual E^α and β-dual E^β of E are defined as $E^\alpha := E^\ell$ and $E^\beta := E^{cs}$, respectively.

If E is a K-space that contains φ, then

$$E^f := \{u_f := (f(e^k)) \mid f \in E'\} \tag{2.3}$$

is called f-dual of E. (Note, throughout we will use the notation $u_p := (p(e^k))$ for each functional $p : E \to \mathbb{K}$.) Moreover, we put

$$\begin{aligned} E_{SAK} &:= \Big\{x \in E \mid \forall f \in E' : f(x) = \sum_k x_k f(e^k)\Big\}, \\ E_{USAK} &:= \Big\{x \in E \mid \forall f \in E' : f(x) = \lim_{\mathcal{F} \in \Phi} \sum_{k \in \mathcal{F}} x_k f(e^k)\Big\}, \end{aligned} \tag{2.4}$$

where Φ is the set of all finite subsets of $\mathbb{N}$ directed by "set inclusion" (cf. [18, 19]). A K-space E is called a *SAK-(USAK-)*space if $E = E_{SAK}$ $(E = E_{USAK})$. If $\bar{\varphi} = E$, then E is an *AD*-space by definition.

Let $A = (a_{nk})$ be an infinite matrix. For a sequence space E we call

$$E_A := \Big\{x \in \omega \mid Ax := \Big(\sum_k a_{nk} x_k\Big)_n \text{ exists and } Ax \in E\Big\} \tag{2.5}$$

domain of A (relative to E). If E is a (separable) *FK*-space, then E_A is too. In particular, the domain $c_A = \{x \in \omega_A \mid \lim_A x := \lim Ax \text{ exists}\}$ is a separable *FK*-space.

Obviously, $\varphi \subset c_A$ if and only if $a_k := \lim_n a_{nk}$ exists for every $k \in \mathbb{N}$. A is called an Sp_1-matrix if $a_k = 1$ $(k \in \mathbb{N})$, and an Sp_1^*-matrix if, in addition, each column of A belongs to bv. If E and F are sequence spaces with $E \subset F_A$, then the linear map $A : E \to F$, $x \to Ax$ is called matrix map.

Let E be a linear space. For a subset M of E^* we use the following notation:

$$\begin{aligned} \overbracket{M} &:= \{g \in E^* \mid \exists (g_n) \text{ in } M \text{ such that } g_n \longrightarrow g(\sigma(E^*, E))\}, \\ \overbracket{\underline{M}} &:= \bigcap \Big\{L \subset E^* \mid L \text{ is a linear subspace of } E^* \text{ and } M \subset L = \overbracket{L}\Big\}, \\ \overbracket{M}^b &:= \{g \in E^* \mid \exists (g_\alpha)_{\alpha \in \mathcal{A}} \text{ in } M \text{ such that} \\ &\qquad \{g_\alpha \mid \alpha \in \mathcal{A}\} \text{ is } \sigma(E^*, E)\text{-bounded and } g_\alpha \longrightarrow g(\sigma(E^*, E))\}, \\ \overbracket{\underline{M}}^b &:= \bigcap \Big\{L \subset E^* \mid L \text{ is a linear subspace of } E^* \text{ and } M \subset L = \overbracket{L}^b\Big\}. \end{aligned} \tag{2.6}$$

Following [5, 6], a K-space E is called an L_φ-space and an A_φ-space, if $E' \subset \overbracket{\underline{\varphi}}$ and $E' \subset \overbracket{\underline{\varphi}}^b$, respectively. Note, $\tau(E, \overbracket{\underline{\varphi}})$ and $\tau(E, \overbracket{\underline{\varphi}}^b)$ is, respectively, the strongest L_φ-topology and A_φ-topology on an arbitrarily given sequence space E.

Theorem 2.1 (see [6, Theorems 3.2 and 3.9]; see also [4, 5]). *Let F be a K-space.*

(a) *F is an L_φ-space if and only if for each Mackey K-space E with $\sigma(E',E)$-sequentially complete dual each matrix map $A : E \to F$ is continuous.*

(b) *F is an A_φ-space if and only if for every barrelled K-space E each matrix map $A : E \to F$ is continuous.*

Theorem 2.2 (see [6, Theorem 4.8]; see also [4]). *Let $A = (a_{nk})$ be a matrix. If E is any L_φ-space (A_φ-space), then E_A (endowed with its natural topology) is an L_φ-space (A_φ-space).*

3. Dual pairs (E, E^S). Throughout, let (S, τ_S) be a K-space containing φ where τ_S is generated by a family $\mathfrak{Q}$ of semi-norms, and, moreover, let $s \in S'$ be a sum on S (cf. [17]), that is,

$$s(z) = \sum_k z_k \quad \text{for each } z \in \varphi. \tag{3.1}$$

Furthermore, let E be a sequence space containing φ. Then (E, E^S) is a dual pair where its bilinear form $\langle \, , \rangle$ is defined by $\langle x, y \rangle := s(yx)$ for all $x \in E$, $y \in E^S$; therefore $E^S \subset E^*$ (up to isomorphy where the isomorphism $E^S \to E^*$ is given by $y \to s \circ \operatorname{diag}_y : E \to \mathbb{K}$ and diag_y is the diagonal matrix (map on E) defined by u). Because of $\varphi \subset E^S$, the weak topology $\sigma(E, E^S)$ is a K-topology. In case of

$$S := cs, \quad s(z) := \sum_k z_k := \lim_n \sum_{k=1}^{n} z_k \quad (z \in cs), \tag{3.2}$$

$$S := \ell, \quad s(z) := \lim_{F \in \Phi} \sum_{k \in F} z_k \quad (z \in \ell), \tag{3.3}$$

we get the dual pairs (E, E^β) and (E, E^α), respectively, which play a fundamental role in summability and the study of topological sequence spaces.

Obviously, $E^\beta \subset E'$ if E is a K-space and $(E', \sigma(E', E))$ is sequentially complete. For example, the latter holds for all barrelled K-spaces.

In view of this remark it is natural to ask for sufficient conditions in order that the inclusion $E^S \subset E'$ holds (up to isomorphy). Aiming an answer to this question we mention (cf. Theorem 2.1) that a matrix map $A : E \to S$ is continuous if one of the following conditions occurs:

(A) S is an L_φ-space, E is a K-space equipped with the Mackey topology $\tau(E, E')$, and $(E', \sigma(E', E))$ is sequentially complete.

(B) S is an A_φ-space and E is a barrelled K-space.

In particular, (A) as well as (B) implies for each $y \in E^S$ the continuity of the matrix map $\operatorname{diag}_y : E \to S$. Thus we have the following proposition.

Proposition 3.1. *If E as well as S enjoy one of the statements (A) or (B), then $E^S \subset E'$.*

Remark 3.2. Let E be a sequence space with $\varphi \subset E$. One may easily check that $(\overleftrightarrow{\varphi}, \sigma(\overleftrightarrow{\varphi}, E))$ is sequentially complete, and $(E, \tau(E, \overleftrightarrow{\varphi}^b))$ is barrelled. As immediate consequences of Proposition 3.1 we obtain that $E^S \subset \overleftrightarrow{\varphi}$ *for each L_φ-space S and* $E^S \subset \overleftrightarrow{\varphi}^b$ *for each A_φ-space S.*

Definition 3.3. For a K-space E containing φ we put

$$E_{SK} := \{x \in E \mid \forall f \in E' : (u_f x) \in S \text{ and } f(x) = s(u_f x)\}. \tag{3.4}$$

E is called an SK-space if $E = E_{SK}$.

Remark 3.4. (a) If E is a K-space containing φ, then $E_{SK} \subset \overline{\varphi}$ in E.
(b) $(E, \tau(E, E^S))$ is an SK-space for each sequence space E with $E \supset \varphi$.

The latter remark is an immediate consequence of the following result which will be useful in the sequel: *Let E be a sequence space containing φ and F be a K-space with $E \subset F$. If the inclusion map $i : (E, \tau(E, E^S)) \to F$ is continuous, then $E \subset F_{SK}$.*

Remark 3.5. (a) In the particular case of (3.2) and (3.3) the SK-property is identical with *SAK* and *USAK*, respectively.
(b) Clearly, if E is an SK-space, then E is an AD-space and $E^f \subset E^S$. Conversely, if (A) or (B) holds, then $E^f \subset E^S$ forces E to be an SK-space. Indeed, in the latter situation $s \circ \operatorname{diag}_{u_f} \in E'$ and the equation $f = s \circ \operatorname{diag}_{u_f}$ extends from φ to E.

4. The solid topology. A sequence space E is solid provided that $yx \in E$ whenever $y \in m$ and $x \in E$. In this situation $x \in E$ if and only if $|x| := (|x_k|) \in E$.
Motivated by Große-Erdmann [12] we introduce some notation.

Notation 4.1. Under the assumption that S is solid for a K-space E containing φ we put

$$E_{SC} := \{x \in E \mid \forall p \in \mathcal{P}_E : u_p x \in S \text{ and } p(x) \le s(u_p |x|)\}, \tag{4.1}$$

where $\mathcal{P}_E$ denotes the family of all continuous semi-norms on E. If $E = E_{SC}$, then E is called an SC-space.
In the particular case of (3.3) we get (cf. [12, page 502])

$$E_{SC} = AC_E := \left\{x \in E \mid \forall p \in \mathcal{P}_E : \sum_k p(x_k e^k) < \infty\right\}. \tag{4.2}$$

We assume throughout this section that S is solid and the sum $s \in S'$ is a positive functional, that is,

$$s(z) \ge 0 \quad \text{for each } z \in S \text{ with } \quad z_k \ge 0 \ (k \in \mathbb{N}). \tag{4.3}$$

An important example for this situation is given in (3.3). We are going to present a further one.

Example 4.2. Let $T = (t_{nk})$ be a normal Sp_1-matrix such that $t_{nk} \ge 0$ $(n, k \in \mathbb{N})$. We put

$$S := \left\{z \in m_T \mid \|z\|_{[T]} := \sup_n \sum_k t_{nk} |z_k| < \infty\right\}. \tag{4.4}$$

As we may easily check, $(S,\|\ \|_{[T]})$ is a solid *BK*-space (containing φ). Now, we will show that there exists a positive sum s on S.

First of all, we note that $(m,\|\ \|_\infty,\geq)$ and $(m_T,\|\ \|_T,\geq_T)$ with $\|\ \|_T := \|\ \|\circ T$ are equivalent as ordered normed spaces, where $z\geq 0$ is defined by $z_k\geq 0$ $(k\in\mathbb{N})$ and $z\geq_T 0$ by $Tz\geq 0$. Since $e:=(1,1,\ldots)$ is an interior point of the positive cone $K:=\{z\in m \mid z\geq 0\}$ in $(m,\|\ \|_\infty)$, we get that $T^{-1}e$ is an interior point of the positive cone $K_T:=\{z\in m_T \mid z\geq_T 0\}$ in $(m_T,\|\ \|_T)$. By a result of Krein (cf. [22, Theorem XIII.2.3]), each positive functional $g\in(c_T,\|\ \|_T)'$ can be extended to a positive continuous linear functional on $(m_T,\|\ \|_T)$. In particular, there exists an $\tilde{s}\in(m_T,\|\ \|_T)'$ such that

$$\tilde{s}|_{c_T}=\lim\nolimits_T,\qquad \tilde{s}(z)\geq 0\quad (z\in K_T). \tag{4.5}$$

Then $s:=\tilde{s}|_S\in(S,\|\ \|_{[T]})'$ and $s(z)\geq 0$ for all $z\in S\cap K_T$. Because of $K\subset K_T$ we have (4.3), and s is a sum since T is an Sp_1-matrix.

Definition 4.3. (a) Let E be a sequence space and G a subspace of E^S containing φ. The locally convex topology $\nu(E,G)$ on E generated by the semi-norms $p_{|y|}$ $(y\in G)$ with

$$p_{|y|}(x):=s(|yx|)\quad (x\in E) \tag{4.6}$$

is called the *solid topology* corresponding to the dual pair (E,G).

(b) Note that $\nu(E,G)$ is the topology $\tau_{\mathfrak{S}}$ of uniform convergence on the solid hulls of $y\in G$, that is, $\nu(E,G)=\tau_{\mathfrak{S}}$ with $\mathfrak{S}:=\{\mathrm{sol}\{y\}\mid y\in G\}$ and $\mathrm{sol}\{y\}:=\{v\in\omega\mid |v_k|\leq|y_k|\ (k\in\mathbb{N})\}$. In fact, it is not difficult to verify that $p_{|y|}(x)=\sup\{|s(vx)|\mid v\in\mathrm{sol}\{y\}\}$. Therefore, $(E,\nu(E,G))'=|G|$ (the solid span of G).

(c) A K-space (E,τ_E) is called simple if each τ_E-bounded subset D of E is dominated by an $x\in E$, that is, $D\subset\mathrm{sol}\{x\}$.

Proposition 4.4. *Let E be a sequence space containing φ.*

(a) *The solid topology $\nu(E,E^S)$ is compatible with (E,E^S), that is, $(E,\nu(E,E^S))'=E^S$.*

(b) *$(E,\nu(E,E^S))$ is an SK-space.*

(c) *$(E,\nu(E,E^S))_{SC}$ is an SC-space.*

Proof. The statement (a) follows from Remark 4.3(b) and the fact that E^S is solid. Obviously, by Remark 3.4(b) we get that (a) implies (b).

(c) Let p be a continuous semi-norm on $(E,\nu(E,E^S))$, then $p(x)=\sup_{y\in V}|s(yx)|$ for each $x\in E$, where $V:=\{y\in E\mid p(y)\leq 1\}^\circ$ (polar in E^S). There exists $v\in E^S$ such that $p(x)\leq p_{|v|}(x)$ for all $x\in E$, hence $p(e^k)\leq|v_k|$ $(k\in\mathbb{N})$. Since E^S is solid and $v\in E^S$, then $u_p=(p(e^k))\in E^S$. Furthermore, $p(e^k)=\sup_{y\in V}|y_k|$ $(k\in\mathbb{N})$ and $|s(yx)|\leq s(|yx|)\leq s(u_p|x|)$ $(x\in E)$ for every $y\in V$, that is, $p(x)\leq s(u_px)$ for each $x\in E$. □

The following proposition is in the particular case of (3.3) a result due to Große-Erdmann (cf. [12, Theorem 3.2] and the erratum in [13]).

Proposition 4.5. *Let E be a solid sequence space containing φ and G be a sequence space with $\varphi\subset G\subset E^S$. Then the following statements are equivalent:*

(a) *$(E,\sigma(E,G))$ is simple.*

(b) $(G,\nu(G,E))$ *is barrelled.*

If in addition G is solid, then (a), thus (b), is equivalent to

(c) $(E,\nu(E,G))$ *is simple.*

Proof. (a)⇒(b). If $(E,\sigma(E,G))$ is simple, then for each $\sigma(E,G)$-bounded subset B of E there exists $y\in E$ such that $B\subset \mathrm{sol}\{y\}$. Thus $\beta(G,E)\subset\nu(G,E)$, which is equivalent to the barrelledness of $(G,\nu(G,E))$.

(b)⇒(a). Suppose B is $\sigma(E,G)$-bounded subset of E. Then the semi-norm p defined by $p(y):=\sup_{x\in B}|s(yx)|$ $(y\in G)$ is continuous on $(G,\beta(G,E))$, which forces $\beta(G,E)=\nu(G,E)$ by (b). Hence $p(y)\le\sup_{x\in\mathrm{sol}\{v\}}|s(yx)|$ $(y\in G)$ for a suitable $v\in E$. Putting $y:=e^k$ $(k\in\mathbb{N})$ we get $\sup_{x\in B}|x_k|=p(e^k)\le\sup_{x\in\mathrm{sol}\{v\}}|x_k|=|v_k|$ $(k\in\mathbb{N})$. Thus, B is dominated by v.

If G is solid, then $\nu(E,G)$- and $\sigma(E,G)$-boundedness of subsets of E are obviously equivalent, thus "(a)⇔(c)" holds. □

5. Quasi-matrix maps. Aiming the extension of some well-known inclusion theorems due to Bennett and Kalton, we consider in the sequel the so-called quasi-matrix maps.

Definition 5.1. Let S be a K-space with a sum $s\in S'$ and let $A=(a_{nk})$ be an infinite matrix and $a^{(n)}$ its nth row. Then the linear map

$$\mathfrak{A}:E\longrightarrow\omega,\qquad x\longmapsto\mathfrak{A}x:=\left(s\left(a^{(n)}x\right)\right)_n,\tag{5.1}$$

where E is a linear subspace of the sequence space $\omega_{\mathfrak{A}}:=\bigcap_{n=1}^{\infty}\{a^{(n)}\}^S$, is called *quasi-matrix map*. Moreover, for every sequence space F the sequence space

$$F_{\mathfrak{A}}:=\left\{x\in\omega_{\mathfrak{A}}\mid\mathfrak{A}x\in F\right\}\tag{5.2}$$

is called a *domain of* $\mathfrak{A}$ *relative to* F and $c_{\mathfrak{A}}$ is called the *domain* of $\mathfrak{A}$; if $x\in c_{\mathfrak{A}}$, we put

$$\lim\nolimits_{\mathfrak{A}}x:=\lim_n s\left(a^{(n)}x\right).\tag{5.3}$$

Note, $\mathfrak{A}x=Ax$ if the matrix A is row-finite.

First of all we give sufficient conditions for the continuity of quasi-matrix maps.

Theorem 5.2. *Let E and F be K-spaces. Each of the following conditions implies the continuity of each quasi-matrix map $\mathfrak{A}:E\to F$:*

(A′) *S and F are L_φ-spaces, E is a Mackey space, and $(E,\sigma(E',E))$ is sequentially complete.*

(B′) *S and F are A_φ-spaces and E is barrelled.*

Proof. Let $\mathfrak{A}:E\to F$ be a quasi-matrix map defined by (5.1). We put $\Delta_{\mathfrak{A}}:=\{f\in F'\mid f\circ\mathfrak{A}\in E'\}$ and show $\Delta_{\mathfrak{A}}=F'$; then $\mathfrak{A}$ is weakly continuous, hence continuous since E carries the Mackey topology $\tau(E,E')$. First of all we note $\varphi\subset\Delta_{\mathfrak{A}}$. Indeed, if $w=(w_1,\dots,w_{n_0},0,\dots)\in\varphi$ and $f(y):=\sum_n w_ny_n$ $(y\in F)$, then we get $f\circ\mathfrak{A}(x)=\sum_{n=1}^{n_0}w_ns(a^{(n)}x)=\sum_{n=1}^{n_0}s(w_na^{(n)}x)=s(vx)$, $x\in E$, where $v:=w_1a^{(1)}+\cdots+w_{n_0}a^{(n)}\in E^S$. Thus $f\circ\mathfrak{A}\in E'$ by Proposition 3.1.

The case (A′). Since $(E', \sigma(E', E))$ is sequentially complete, then $\overbracket{\Delta_{\mathfrak{A}}} \cap F' = \Delta_{\mathfrak{A}}$. Therefore, the fact that F is an L_φ-space implies $F' = \overbracket{\varphi} \cap F' \subset \overbracket{\Delta_{\mathfrak{A}}} \cap F' = \Delta_{\mathfrak{A}}$.

The case (B′). The barrelledness of E implies $\overbracket{\Delta_{\mathfrak{A}}}^{b} \cap F' = \Delta_{\mathfrak{A}}$. Since F is an A_φ-space, we have $F' = \overbracket{\varphi}^{b} \cap F' \subset \overbracket{\Delta_{\mathfrak{A}}}^{b} \cap F' = \Delta_{\mathfrak{A}}$. □

Now, we are going to define a K-topology on the domain $E_{\mathfrak{A}}$, where E is a K-space topologized by a family $\mathcal{P}$ of semi-norms and $\mathfrak{A}$ is a quasi-matrix map defined by (5.1). The set $\mathfrak{Q}_n := \{r_k \mid k \in \mathbb{N}\} \cup \{q \circ \operatorname{diag}_{a^{(n)}} \mid q \in \mathcal{Q}\}$ of semi-norms generates a K-topology on $\{a^{(n)}\}^S$ $(n \in \mathbb{N})$. If S is an L_φ- or A_φ-space, then the K-space $\{a^{(n)}\}^S$ enjoys the same property. Clearly, the family $\mathfrak{Q} := \bigcup_{n \in \mathbb{N}} \mathfrak{Q}_n$ generates a K-topology $\tau_{\omega_{\mathfrak{A}}}$ on $\omega_{\mathfrak{A}}$. By the following lemma, the L_φ- and A_φ-property of the K-space S implies the same property on $(\omega_{\mathfrak{A}}, \tau_{\omega_{\mathfrak{A}}})$.

Lemma 5.3. *Let S be an L_φ-space (A_φ-space) and let $\{F_\alpha \mid \alpha \in \mathcal{A}\}$ be a family of L_φ-spaces (A_φ-spaces). Then $\mathcal{R} := \bigcup_{\alpha \in \mathcal{A}} \mathcal{R}_\alpha$, where $\mathcal{R}_\alpha$ is a family of semi-norms on F_α $(\alpha \in \mathcal{A})$ generating the topology of F_α, generates an L_φ-topology (A_φ-topology) on $F := \bigcap_{\alpha \in \mathcal{A}} F_\alpha$.*

Proof. First assume that S and F_α $(\alpha \in \mathcal{A})$ are L_φ-spaces and (E, τ_E) is a Mackey K-space such that $(E', \sigma(E', E))$ is sequentially complete. Moreover, let $A : E \to F$ be a matrix map. Because of Theorem 2.1(a) the map $A : E \to F_\alpha$ is continuous for each $\alpha \in \mathcal{A}$. This implies the continuity of $A : E \to F$. By Theorem 2.1(a), F is an L_φ-space.

For the case of A_φ-spaces we use Theorem 2.1(b). □

Proposition 5.4. *If E and S are L_φ-spaces (A_φ-spaces), then $(E_{\mathfrak{A}}, \tau_{E_{\mathfrak{A}}})$ topologized by the family $\mathfrak{Q} \cup \{p \circ \mathfrak{A} \mid p \in \mathcal{P}\}$ of semi-norms is also an L_φ-space (A_φ-space).*

Proof. Obviously, $E_{\mathfrak{A}}$ is a K-space. First of all we verify that $(E_{\mathfrak{A}}, \tau_{E_{\mathfrak{A}}})$ is an L_φ-space: if E and S are L_φ-spaces, then $\omega_{\mathfrak{A}}$ is also an L_φ-space. By Theorem 5.2, the maps

$$i_\omega : (E_{\mathfrak{A}}, \tau(E_{\mathfrak{A}}, \overbracket{\varphi})) \longrightarrow \omega_{\mathfrak{A}}, \quad x \longmapsto x, \qquad \mathfrak{A} : (E_{\mathfrak{A}}, \tau(E_{\mathfrak{A}}, \overbracket{\varphi})) \longrightarrow E, \quad x \longmapsto \mathfrak{A}x, \tag{5.4}$$

are continuous. This implies the continuity of the identity map $i : (E_{\mathfrak{A}}, \tau(E_{\mathfrak{A}}, \overbracket{\varphi})) \to (E_{\mathfrak{A}}, \tau_{E_{\mathfrak{A}}})$. Since $(E, \tau(E_{\mathfrak{A}}, \overbracket{\varphi}))$ is an L_φ-space we get that $(E_{\mathfrak{A}}, \tau_{E_{\mathfrak{A}}})$ is an L_φ-space too.

For the case of A_φ-spaces we use the fact that $(E_{\mathfrak{A}}, \tau(E_{\mathfrak{A}}, \overbracket{\varphi}^{b}))$ is an A_φ-space. □

Remark 5.5. If E is a separable FK-space and S is a separable BK-space, then $(E_{\mathfrak{A}}, \tau_{E_{\mathfrak{A}}})$ is a separable FK-space. The proof of this fact is similar to the one of [1, Theorem 1].

By Proposition 5.4, the domain $c_{\mathfrak{A}}$ of a quasi-matrix map $\mathfrak{A}$ is an L_φ- or A_φ-space if S has the same property. If $a_k := \lim_n a_{nk}$ exists, then $\varphi \subset c_{\mathfrak{A}}$, and we put $a := (a_k)$ and define

$$\Lambda_{\mathfrak{A}}^{S\perp} := \{x \in c_{\mathfrak{A}} \mid ax \in S \text{ and } \lim_{\mathfrak{A}} x = s(ax)\}. \tag{5.5}$$

Note, because of $\lim_{\mathfrak{A}} \in c'_{\mathfrak{A}}$, we have $(c_{\mathfrak{A}})_{SK} \subset \Lambda_{\mathfrak{A}}^{S\perp}$.

6. Inclusion theorems of Bennett-Kalton type. In this section, we extend well-known inclusion theorems of Bennett and Kalton [3, Theorems 4 and 5] (cf. also authors' results [5, Theorem 4.4] and [6, Theorem 5.1]) to the situation considered above. We extend moreover a theorem due to Große-Erdmann (cf. [12]) which is of the same type as the mentioned theorems of Bennett and Kalton. There are no new ideas in the proofs, however for the sake of completeness we give them in a brief form.

Theorem 6.1. *Let S be an L_φ-space. For any sequence space E containing φ the following statements are equivalent:*

(a) $(E^S,\sigma(E^S,E))$ *is sequentially complete.*

(b) *Each quasi-matrix map* $\mathfrak{A}:(E,\tau(E,E^S))\to F$ *is continuous when F is an L_φ-space.*

(c) *The inclusion map* $i:(E,\tau(E,E^S))\to F$ *is continuous whenever F is an L_φ-space containing E.*

(d) *The implication* $E\subset F\Rightarrow E\subset F_{SK}$ *holds whenever F is an L_φ-space.*

(e) *The implication* $E\subset c_{\mathfrak{A}}\Rightarrow E\subset \Lambda_{\mathfrak{A}}^{S\perp}$ *holds for every quasi-matrix map $\mathfrak{A}$.*

Proof. (a)⇒(b) is an immediate consequence of Theorem 5.2(A′), (b)⇒(c) is obviously valid whereas (c)⇒(d) follows from Remark 3.4(b). Furthermore, "(d)⇒(e)" is true since $c_{\mathfrak{A}}$ is an L_φ-space and $(c_{\mathfrak{A}})_{SK}\subset\Lambda_{\mathfrak{A}}^{S\perp}$.

(e)⇒(a). If $(a^{(n)})$ is a Cauchy sequence in $(E^S,\sigma(E^S,E))$, then $E\subset c_{\mathfrak{A}}$, where the quasi-matrix map $\mathfrak{A}$ is defined by the matrix A with $a^{(n)}$ as nth row. On account of (e) we get $E\subset\Lambda_{\mathfrak{A}}^{S\perp}$, thus $a\in E^S$ and $a^{(n)}\to a(\sigma(E^S,E))$. □

Theorem 6.2. *Let S be an A_φ-space. For any sequence space E containing φ the following statements are equivalent:*

(a) $(E,\tau(E,E^S))$ *is barrelled.*

(b) *Each quasi-matrix map* $\mathfrak{A}:(E,\tau(E,E^s))\to F$ *is continuous when F is an A_φ-space.*

(c) *The inclusion map* $i:(E,\tau(E,E^S))\to F$ *is continuous whenever F is an A_φ-space containing E.*

(d) *The implication* $E\subset F\Rightarrow E\subset F_{SK}$ *holds whenever F is an A_φ-space.*

(e) *The implication* $E\subset m_{\mathfrak{A}}\Rightarrow E\subset\Lambda_{\mathfrak{A}}^{S\perp}$ *holds for every quasi-matrix map $\mathfrak{A}$ with* $\varphi\subset c_{\mathfrak{A}}$.

(f) *Every* $\sigma(E^S,E)$*-bounded subset of E^S is relatively sequentially* $\sigma(E^S,E)$*-compact.*

Proof. (a)⇒(b) is an immediate consequence of Theorem 5.2(B′), (b)⇒(c) is obviously valid. (c)⇒(d) follows from Remark 3.4(b). (d)⇒(e) is true, since $m_{\mathfrak{A}}$ is an A_φ-space (cf. Proposition 5.4) and $(m_{\mathfrak{A}})_{SK}=(c_{\mathfrak{A}})_{SK}\subset\Lambda_{\mathfrak{A}}^{S\perp}$.

(e)⇒(f). Let B be a $\sigma(E^S,E)$-bounded subset of E^S and let $(b^{(r)})$ be a sequence in B. Obviously, $(b^{(r)})$ is bounded in (ω,τ_ω), thus we may choose a coordinatewise convergent subsequence $(a^{(n)})$ of $(b^{(r)})$. Because of the $\sigma(E^S,E)$-boundedness of $\{a^{(n)}\mid n\in\mathbb{N}\}$ we get $E\subset m_{\mathfrak{A}}$, where $\mathfrak{A}$ is the quasi-matrix map defined by the matrix $A=(a_k^{(n)})_{n,k}$. Applying (e) we obtain $E\subset\Lambda_{\mathfrak{A}}^{S\perp}$, thus $a\in E^S$ and $a^{(n)}\to a(\sigma(E^S,E))$. Hence B is relatively sequentially compact in $(E^S,\sigma(E^S,E))$.

(f)⇒(a). Using the fact that in the K-space $(E^S,\sigma(E^S,E))$ each relatively sequentially compact subset is relatively compact too (cf. [14, Theorem 3.11, page 61]), (f) tells us that $(E,\tau(E,E^S))$ is barrelled. □

Motivated by Große-Erdmann [12, Theorem 4.1], we prove next an inclusion theorem which gives a connection between the barrelledness of $(E,\nu(E,E^S))$ and the implication $E\subset F \Rightarrow E\subset F_{SC}$ where F is an A_φ-space.

Theorem 6.3. *Let S be a solid A_φ-space with a positive sum $s\in S'$. Let E be any sequence space containing φ. The following statements are equivalent:*

(a) *$(E,\nu(E,E^S))$ is barrelled.*

(b) *$(E^S,\sigma(E^S,E))$ is simple.*

(c) *Each quasi-matrix map $\mathfrak{A}:(E,\nu(E,E^S))\to F$ is continuous when F is an A_φ-space.*

(d) *The inclusion map $i:(E,\nu(E,E^S))\to F$ is continuous whenever F is an A_φ-space containing E.*

(e) *The implication $E\subset F\Rightarrow E\subset F_{SC}$ holds whenever F is an A_φ-space.*

Proof. (a)⇔(b) is a part of Proposition 4.5, (a)⇒(c) is an immediate consequence of Theorem 5.2(B′), (c)⇒(d) is obviously true, and (d)⇒(e) follows from Proposition 4.4(c).

(e)⇒(d). For every $p\in\mathcal{P}_F$ we get $u_p\in E^S$ and $p(x)\le p_{|u_p|}(x)$ $(x\in E)$. Thus $p|_E$ is continuous on $(E,\nu(E,E^S))$. Consequently, $i:(E,\nu(E,E^S))\to F$ is continuous.

(d)⇒(a). First of all we remark that $(E,\nu(E,E^S))$ is separable as an *SK*-space. Thus, we have to show that it is ω-barrelled, that is, that every countable $\sigma(E^S,E)$-bounded subset $\{a^{(n)}\mid n\in\mathbb{N}\}$ of E^S is $\nu(E,E^S)$-equicontinuous (cf. [14, page 27, Theorem 10.2]). For a proof of this we consider the quasi-matrix map $\mathfrak{A}$ defined by the matrix $A=(a_k^{(n)})_{n,k}$. Then $E\subset m_{\mathfrak{A}}$ since A is $\sigma(E^S,E)$-bounded, and $(m_{\mathfrak{A}},\tau_{\mathfrak{A}})$ is an A_φ-space because of Proposition 5.4. Hence $i:(E,\nu(E,E^S))\to(m_{\mathfrak{A}},\tau_{\mathfrak{A}})$ is continuous. Furthermore, the quasi-matrix map $\mathfrak{A}:(m_{\mathfrak{A}},\tau_{m_{\mathfrak{A}}})\to(m,\|\;\|_\infty)$ is continuous since $\|\;\|_\infty\circ\mathfrak{A}$ is a continuous semi-norm on $m_{\mathfrak{A}}$. Altogether, $\mathfrak{A}:(E,\nu(E,E^S))\to(m,\|\;\|_\infty)$ is also continuous. Thus there exists $v\in E^S$ such that $\|\mathfrak{A}x\|_\infty=\sup_n|s(a^{(n)}x)|\le p_{|v|}(x)$ for each $x\in E$. Consequently, the equicontinuity of $\{a^{(n)}\mid n\in\mathbb{N}\}$ is proved. □

7. Applications. In this section, we deal with applications of our general inclusion theorems to the case of certain dual pairs (E,E^S). In various situations we discuss connections between weak sequential completeness and barrelledness on one hand and certain modifications of weak sectional convergence on the other hand.

Case 7.1 ($S=c_T$ and $S=cs$). Let $T=(t_{nk})$ be a fixed row-finite Sp_1-matrix and

$$S:=c_T \quad \text{as well as} \quad \lim\nolimits_T s(z):=z \quad (z\in c_T). \tag{7.1}$$

Clearly, S endowed with its (separable) *FK*-topology is an L_φ- and A_φ-space. We have (cf. [8, 15])

$$E^S=E^{\beta(T)}:=\Big\{y\in\omega\mid \forall x\in E:\lim_n\sum_k t_{nk}y_kx_k \text{ exists}\Big\} \tag{7.2}$$

for each sequence space E and

$$E_{SK}=E_{STK}:=\Big\{x\in E\mid \forall f\in E':f(x)=\lim_n\sum_k t_{nk}x_kf(e^k)\Big\} \tag{7.3}$$

(weak T-sectional convergence) if E is a K-space containing φ.

Aiming a characterization of quasi-matrix maps in this context we need the following concept (cf. [7]): let $\mathscr{V}$ be a double sequence space, that is a linear subspace of the linear space Ω of all double sequences $y = (y_{nr})$, and let $\mathscr{A} = (A^{(r)})$ be a sequence of infinite matrices $A^{(r)} = (a_{nk}^{(r)})_{n,k}$. We put

$$\Omega_{\mathscr{A}} := \bigcap_r \omega_{A^{(r)}}, \qquad \mathscr{V}_{\mathscr{A}} := \left\{ x \in \Omega_{\mathscr{A}} \mid \mathscr{A}x := \left(\sum_k a_{nk}^{(r)} x_k \right)_{nr} \in \mathscr{V} \right\}. \tag{7.4}$$

If there exists a limit functional on $\mathscr{V}$, say $\mathscr{V}$-lim, then the summability method induced by $\mathscr{V}_{\mathscr{A}}$ and the limit functional

$$\mathscr{V}\text{-}\lim_{\mathscr{A}} : \mathscr{V}_{\mathscr{A}} \longrightarrow \mathbb{K}, \qquad x \longmapsto \mathscr{V}\text{-}\lim \mathscr{A}x, \tag{7.5}$$

is called a $\mathscr{V}$-*SM*-method. For example, this definition contains as a special case the $\mathscr{C}_c$-*SM*-methods considered by Przybylski [16] where

$$\mathscr{C}_c := \left\{ y = (y_{nr})_{n,r\in\mathbb{N}} \;\middle|\; \lim_r \lim_n y_{nr} =: \mathscr{C}_c\text{-}\lim y \text{ exists} \right\}. \tag{7.6}$$

Furthermore, if

$$\mathscr{C}_m := \left\{ (y_{nr}) \in \Omega \mid \forall r \in \mathbb{N} : (y_{nr})_n \in c \text{ and } \left(\lim_n y_{nr} \right)_r \in m \right\} \tag{7.7}$$

and $\mathscr{A} = (A^{(r)})$ is any sequence of matrices, then $\mathscr{C}_{m\mathscr{A}} := \{x \in \Omega_{\mathscr{A}} \mid \mathscr{A}x \in \mathscr{C}_m\}$ is an *FK*-space and $\mathscr{C}_{c\mathscr{A}}$ is a closed separable subspace of $\mathscr{C}_{m\mathscr{A}}$.

The introduced notation enables us to describe quasi-matrix maps in the situation of (7.1): if $A = (a_{nk})$ is any matrix, then the domain $c_{\mathfrak{A}}$ of the corresponding quasi-matrix map $\mathfrak{A}$ is precisely the domain $\mathscr{C}_{c\mathscr{A}}$ of the $\mathscr{C}_c$-*SM*-method $\mathscr{A} = (A^{(r)})$ with $A^{(r)} := (t_{nk}a_{rk})_{n,k}$ $(r \in \mathbb{N})$. Thus $m_{\mathfrak{A}} = \mathscr{C}_{m\mathscr{A}}$ is an A_φ-space and $c_{\mathfrak{A}} = \mathscr{C}_{c\mathscr{A}}$ is an L_φ- and A_φ-space (where we consider them as *FK*-spaces).

As consequences of the main Theorems 6.1 and 6.2 we get the following inclusion theorems in the situation of the dual pair $(E, E^{\beta(T)})$.

Theorem 7.2. *For any sequence space E with $\varphi \subset E$ the following statements are equivalent:*

(a) $(E^{\beta(T)}, \sigma(E^{\beta(T)}, E))$ *is sequentially complete.*

(b) *The implication $E \subset F \Rightarrow E \subset F_{STK}$ holds whenever F is an L_φ-space.*

(c) *The implication $E \subset \mathscr{C}_{c\mathscr{A}} \Rightarrow E \subset (\mathscr{C}_{c\mathscr{A}})_{STK}$ holds for each sequence $\mathscr{A}$ of matrices.*

Theorem 7.3. *For any sequence space E with $\varphi \subset E$ the following statements are equivalent:*

(a) $(E, \tau(E, E^{\beta(T)}))$ *is barrelled.*

(b) *The implication $E \subset F \Rightarrow E \subset F_{STK}$ holds whenever F is an A_φ-space.*

(c) *The implication $E \subset \mathscr{C}_{m\mathscr{A}} \Rightarrow E \subset (\mathscr{C}_{c\mathscr{A}})_{STK}$ holds for each sequence $\mathscr{A}$ of matrices satisfying $\varphi \subset \mathscr{C}_{c\mathscr{A}}$.*

Note, in the particular case of $T := \Sigma$ (summation matrix) we get the well-known inclusion theorems of Bennett and Kalton [3, Theorems 4 and 5] in a generalized version due to the authors (cf. [5, Theorem 4.4] and [6, Theorem 5.1]).

Case 7.4 ($S = bv_T$). Let $T = (t_{nk})$ be a fixed row-finite Sp_1^*-matrix and

$$S := bv_T \quad \text{as well as} \quad s(z) := \lim_{\mathcal{F}\in\Phi} \sum_{n\in\mathcal{F}} \sum_k (t_{nk} - t_{n-1,k}) z_k = \lim_T z \quad (z \in bv_T). \tag{7.8}$$

Then S equipped with its (separable) *FK*-topology is an L_φ- and A_φ-space. We have (cf. [10, 11])

$$E^S = E^{\alpha(T)} := \left\{ y \in \omega \;\middle|\; \sum_n \left| \sum_k (t_{nk} - t_{n-1,k}) y_k x_k \right| < \infty \right\} \tag{7.9}$$

for any sequence space E. Moreover, if E is a K-space containing φ, then we obtain

$$E_{SK} = E_{USTK} := \left\{ x \in E \mid \forall f \in E' : \sum_n \left| \sum_k (t_{nk} - t_{n-1,k}) x_k f(e^k) \right| < \infty, \right. \\ \left. f(x) = \lim_n \sum_k t_{nk} x_k f(e^k) \right\} \tag{7.10}$$

(cf. [11, Theorem 3.1]).

For a description of quasi-matrix maps in the situation of (7.8) we introduce the double sequence spaces

$$\begin{aligned} \mathcal{B}v_m &:= \left\{ (y_{nr}) \mid \forall r \in \mathbb{N} : (y_{nr})_n \in bv \text{ and } \left(\lim_n y_{nr} \right)_r \in m \right\}, \\ \mathcal{B}v_c &:= \left\{ (y_{nr}) \mid \forall r \in \mathbb{N} : (y_{nr})_n \in bv \text{ and } \left(\lim_n y_{nr} \right)_r \in c \right\}, \end{aligned} \tag{7.11}$$

and the limit functional $\mathcal{B}v_c\text{-}\lim : \mathcal{B}v_c \to \mathbb{K}$, $(y_{nr}) \mapsto \lim_r \lim_n y_{nr}$. It is a standard exercise to prove that $\mathcal{B}v_{m_{\mathcal{A}}}$ is an *FK*-space for each sequence $\mathcal{A} = (A^{(r)})$ of matrices and that $\mathcal{B}v_{c_{\mathcal{A}}}$ is a closed separable subspace of $\mathcal{B}v_{m_{\mathcal{A}}}$.

For a matrix $A = (a_{nk})$ the corresponding quasi-matrix map $\mathfrak{A}$ in the context of (7.8) is the $\mathcal{B}v_c$-*SM*-method $\mathcal{A} = (A^{(r)})$ with $A^{(r)} := (t_{nk} a_{rk})_{nk}$ $(r \in \mathbb{N})$. Thus $m_{\mathfrak{A}} = \mathcal{B}v_{m_{\mathcal{A}}}$ and $c_{\mathfrak{A}} = \mathcal{B}v_{c_{\mathcal{A}}}$.

Applying the main Theorems 6.1 and 6.2 to the situation of (7.8), we get the following inclusion theorems.

Theorem 7.5. *For any sequence space E with $\varphi \subset E$ the following statements are equivalent:*

(a) $(E^{\alpha(T)}, \sigma(E^{\alpha(T)}, E))$ *is sequentially complete.*

(b) *The implication $E \subset F \Rightarrow E \subset F_{USTK}$ holds whenever F is an L_φ-space.*

(c) *The implication $E \subset \mathcal{B}v_{c_{\mathcal{A}}} \Rightarrow E \subset (\mathcal{B}v_{c_{\mathcal{A}}})_{USTK}$ holds for each sequence $\mathcal{A}$ of matrices.*

Theorem 7.6. *For any sequence space E with $\varphi \subset E$ the following statements are equivalent:*

(a) $(E, \tau(E, E^{\alpha(T)}))$ *is barrelled.*

(b) *The implication $E \subset F \Rightarrow E \subset F_{USTK}$ holds whenever F is an A_φ-space.*

(c) *The implication $E \subset \mathcal{B}v_{m_{\mathcal{A}}} \Rightarrow E \subset (\mathcal{B}v_{c_{\mathcal{A}}})_{USTK}$ holds for each sequence $\mathcal{A}$ of matrices satisfying $\varphi \subset \mathcal{B}v_{c_{\mathcal{A}}}$.*

Case 7.7 ($S = \ell$)**.** We consider the case of (3.3). Clearly, this is a particular case of (7.8) with $T = \Sigma$. Then $E^S = E^\alpha$ and $E_{SK} = E_{USAK}$ (cf. [18, 19]). A quasi-matrix map induced by a matrix $A = (a_{nk})$ is in this situation a matrix map $y_n = \sum_k a_{nk}x_k$ $(n \in \mathbb{N})$ with the application domain $\omega_{|A|} := \{x \in \omega \mid \forall n \in \mathbb{N} : \sum_k |a_{nk}x_k| < \infty\}$. We now put $m_{|A|} := m_A \cap \omega_{|A|}$, $c_{|A|} := c_A \cap \omega_{|A|}$ and $\Lambda^{\perp}_{|A|} := \{x \in c_{|A|} \mid \sum_k |a_k x_k| < \infty$ and $\lim_A x = \sum_k a_k x_k\}$ (here we assume $\varphi \subset c_A$). From Theorems 6.1 and 6.2 we get the following inclusion theorems.

Theorem 7.8. *For any sequence space E with $\varphi \subset E$ the following statements are equivalent:*

(a) $(E^\alpha, \sigma(E^\alpha, E))$ *is sequentially complete.*

(b) *Every matrix map*

$$A : (E, \tau(E, E^\alpha)) \longrightarrow F \quad \text{with } E \subset \omega_{|A|} \tag{7.12}$$

is continuous whenever F is an L_φ-space.

(c) *The implication $E \subset F \Rightarrow E \subset F_{USAK}$ holds whenever F is an L_φ-space.*

(d) *The implication $E \subset c_{|A|} \Rightarrow E \subset \Lambda^{\perp}_{|A|}$ holds for each matrix A.*

Note that Bennett [2] proved, that $(E^\alpha, \sigma(E^\alpha, E))$ is sequentially complete if E is a monotone sequence space, and that Swartz and Stuart [21, Theorem 5] (see also [20, Theorem 7]) gave a more general class of sequence spaces having that property.

Theorem 7.9. *For any sequence space E with $\varphi \subset E$ the following statements are equivalent:*

(a) $(E, \tau(E, E^\alpha))$ *is barrelled.*

(b) *Any matrix map according to (7.12) is continuous when F is an A_φ-space.*

(c) *The implication $E \subset F \Rightarrow E \subset F_{USAK}$ holds whenever F is an A_φ-space.*

(d) *The implication $E \subset m_{|A|} \Rightarrow E \subset \Lambda^{\perp}_{|A|}$ holds for each matrix A satisfying $\varphi \subset c_A$.*

Case 7.10 ($S = [cs]$)**.** Let

$$S := [cs] := \left\{ z \in cs \;\middle|\; \lim_j \sum\nolimits_{2^j} |z_k| = 0 \right\}, \quad s(z) := \sum_k z_k \quad (z \in [cs]), \tag{7.13}$$

where $\sum_{2^j} a_k := \sum_{k=2^j}^{2^{j+1}-1} a_k$ (cf. [9]). It is known that $[cs]$ is an AK-BK-space with the norm $\|\ \|$ defined by $\|z\| := \sup_m |\sum_{k=1}^m z_k| + \sup_j \sum_{2^j} |z_k|$ $(z \in [cs])$ (cf. [9, Theorem 4.5]). Thus, $[cs]$ is an L_φ- and A_φ-space. For a sequence space E we set $E^{[\beta]} := E^{[cs]} = \{y \in E^\beta \mid \lim_j \sum_{2^j} |y_k x_k| = 0\}$. If E is a K-space containing φ, we get

$$\begin{aligned} E_{SK} &= \left\{ x \in E \mid \forall f \in E' : (u_f x) \in [cs] \text{ and } f(x) = \sum_k x_k f(e^k) \right\} \\ &= \left(E^f\right)^{[\beta]} \cap E_{SAK}. \end{aligned} \tag{7.14}$$

The quasi-matrix map $\mathfrak{A} : \omega_{\mathfrak{A}} \to \omega$ corresponding to a matrix $A = (a_{nk})$ is precisely the matrix map $A : \omega_{[A]} \to \omega$ with

$$\omega_{[A]} := \left\{ x \in \omega_A \mid \forall n \in \mathbb{N} : \lim_j \sum_{2^j} |a_{nk}x_k| = 0 \right\} = \omega_{\mathfrak{A}}. \tag{7.15}$$

We put $m_{[A]} := m_A \cap \omega_{[A]}$, $c_{[A]} := c_A \cap \omega_{[A]}$ and, assuming $\varphi \subset c_A$,

$$\Lambda^{\perp}_{[A]} := \left\{ x \in c_{[A]} \mid ax \in [cs] \text{ and } \lim_A x = \sum_k a_k x_k \right\}. \tag{7.16}$$

From Theorems 6.1 and 6.2 we obtain the following inclusion theorems in the context of (7.13).

Theorem 7.11. *For any sequence space E with $\varphi \subset E$ the following statements are equivalent:*

(a) $(E^{[\beta]}, \sigma(E^{[\beta]}, E))$ *is sequentially complete.*

(b) *Each matrix map*

$$A : (E, \tau(E, E^{[\beta]})) \longrightarrow F \quad \text{with } E \subset \omega_{[A]} \tag{7.17}$$

is continuous whenever F is an L_φ-space.

(c) *The implication $E \subset F \Rightarrow E \subset (F^f)^{[\beta]} \cap F_{SAK}$ holds whenever F is an L_φ-space.*

(d) *The implication $E \subset c_{[A]} \Rightarrow E \subset \Lambda^{\perp}_{[A]}$ holds for each matrix A.*

Theorem 7.12. *For any sequence space E with $\varphi \subset E$ the following statements are equivalent:*

(a) $(E, \tau(E, E^{[\beta]}))$ *is barrelled.*

(b) *Each matrix map according to (7.17) is continuous when F is an A_φ-space.*

(c) *The implication $E \subset F \Rightarrow E \subset (F^f)^{[\beta]} \cap F_{SAK}$ holds whenever F is an A_φ-space.*

(d) *The implication $E \subset m_{[A]} \Rightarrow E \subset \Lambda^{\perp}_{[A]}$ holds for each matrix A.*

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References

[1] G. Bennett, *A representation theorem for summability domains*, Proc. London Math. Soc. (3) **24** (1972), 193–203. MR 45#776. Zbl 0229.40007.

[2] ———, *A new class of sequence spaces with applications in summability theory*, J. Reine Angew. Math. **266** (1974), 49–75. MR 49#9585. Zbl 0277.46012.

[3] G. Bennett and N. J. Kalton, *Inclusion theorems for K-spaces*, Canad. J. Math. **25** (1973), 511–524. MR 48#836. Zbl 0272.46009.

[4] J. Boos, K.-G. Große-Erdmann, and T. Leiger, *L_ϕ-spaces and some related sequence spaces*, Z. Anal. Anwendungen **13** (1994), no. 3, 377–385. MR 95g:46013. Zbl 0807.46007.

[5] J. Boos and T. Leiger, *Some new classes in topological sequence spaces related to L_r-spaces and an inclusion theorem for $K(X)$-spaces*, Z. Anal. Anwendungen **12** (1993), no. 1, 13–26. MR 94i:40010. Zbl 0781.46014.

[6] ———, *"Restricted" closed graph theorems*, Z. Anal. Anwendungen **16** (1997), no. 3, 503–518. MR 99a:46007. Zbl 0926.46003.

[7] J. Boos, T. Leiger, and K. Zeller, *Consistency theory for SM-methods*, Acta Math. Hungar. **76** (1997), no. 1-2, 109–142. MR 99i:40006. Zbl 0930.40001.

[8] M. Buntinas, *On Toeplitz sections in sequence spaces*, Math. Proc. Cambridge Philos. Soc. **78** (1975), no. 3, 451–460. MR 53#13913. Zbl 0314.46012.

[9] M. Buntinas and N. Tanović-Miller, *Strong boundedness and strong convergence in sequence spaces*, Canad. J. Math. **43** (1991), no. 5, 960–974. MR 92m:46012. Zbl 0790.46003.

[10] J. DeFranza and D. J. Fleming, *Sequence spaces and summability factors*, Math. Z. **199** (1988), no. 1, 99–108. MR 90c:40011. Zbl 0631.40008.

[11] D. J. Fleming, *Unconditional Toeplitz sections in sequence spaces*, Math. Z. **194** (1987), no. 3, 405–414. MR 88h:46010. Zbl 0596.40006.

[12] K.-G. Große-Erdmann, *Matrix transformations involving analytic sequence spaces*, Math. Z. **209** (1992), no. 4, 499–510. MR 93d:46018. Zbl 0782.46012.

[13] ———, *Errata: "Matrix transformations involving analytic sequence spaces"*, Math. Z. **212** (1993), no. 1, 143, [Math. Z. **209** (1992) 499–510. MR 93d:46018]. MR 93k:46006. Zbl 0833.46003.

[14] P. K. Kamthan and M. Gupta, *Sequence Spaces and Series*, Lecture Notes in Pure and Applied Mathematics, vol. 65, Marcel Dekker, New York, 1981. MR 83g:46011. Zbl 0447.46002.

[15] G. Meyers, *On Toeplitz sections in FK-spaces*, Studia Math. **51** (1974), 23–33. MR 50#943. Zbl 0275.46007.

[16] B. Przybylski, *On the perfectness of methods defined by the iteration product of matrix transformations*, Ph.D. thesis, University of Łódź, 1977.

[17] W. H. Ruckle, *An abstract concept of the sum of a numerical series*, Canad. J. Math. **22** (1970), 863–874. MR 42#3463. Zbl 0201.15702.

[18] J. Sember and M. Raphael, *The unrestricted section properties of sequences*, Canad. J. Math. **31** (1979), no. 2, 331–336. MR 80j:46021. Zbl 0396.46006.

[19] J. J. Sember, *On unconditional section boundedness in sequence spaces*, Rocky Mountain J. Math. **7** (1977), no. 4, 699–706. MR 56#6338. Zbl 0378.46009.

[20] C. Swartz, *Infinite Matrices and the Gliding Hump*, World Scientific Publishing, New Jersey, 1996. MR 98b:46002. Zbl 0923.46003.

[21] C. Swartz and C. Stuart, *A projection property and weak sequential completeness of α-duals*, Collect. Math. **43** (1992), no. 2, 177–185. MR 94h:40014. Zbl 0779.46013.

[22] B. Z. Vulih, *Vvedenie v teoriyu poluuporyadochennykh prostranstv* [*Introduction to the Theory of Partially Ordered Spaces*], Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1961 (Russian). MR 24#A3494.

[23] A. Wilansky, *Modern Methods in Topological Vector Spaces*, McGraw-Hill, New York, 1978. MR 81d:46001. Zbl 0395.46001.

[24] ———, *Summability through Functional Analysis*, North-Holland Mathematics Studies, vol. 85, North-Holland, Amsterdam, 1984. MR 85d:40006. Zbl 0531.40008.

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