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ON ZERO SUBRINGS AND PERIODIC SUBRINGS

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ABSTRACT. We give new proofs of two theorems on rings in which every zero subring is finite; and we apply these theorems to obtain a necessary and sufficient condition for an infinite ring with periodic additive group to have an infinite periodic subring.

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Let R be a ring and N its set of nilpotent elements; and call R reduced if $N = \{0\}$. Following [4], call R an *FZS*-ring if every zero subring—that is, every subring with trivial multiplication—is finite. It was proved in [1] that every nil *FZS*-ring is finite—a result which in more transparent form is as follows.

THEOREM 1. *Every infinite nil ring contains an infinite zero subring.*

Later, in [4], it was shown that every ring with N infinite contains an infinite zero subring. The proof relies on Theorem 1 together with the following result.

THEOREM 2 (see [4])**.** *If R is any semiprime FZS-ring, then $R = B \oplus C$, where B is reduced and C is a direct sum of finitely many total matrix rings over finite fields.*

Theorems 1 and 2 have had several applications in the study of commutativity and finiteness. Since the proofs in [1, 4] are rather complicated, it is desirable to have new and simpler proofs; and in our first major section, we present such proofs. In our final section, we apply Theorems 1 and 2 in proving a new theorem on existence of infinite periodic subrings.

1. Preliminaries. Let $\mathbb{Z}$ and $\mathbb{Z}^+$ denote, respectively the ring of integers and the set of positive integers. For the ring R, denote by the symbols T and $P(R)$, respectively the ideal of torsion elements and the prime radical; and for each $n \in \mathbb{Z}^+$, define R_n to be $\{x \in R \mid x^n = 0\}$. For Y an element or subset of R, let $\langle Y \rangle$ be the subring generated by Y; let $A_l(Y)$, $A_r(Y)$, and $A(Y)$ be the left, right, and two-sided annihilators of Y; and let $C_R(Y)$ be the centralizer of Y. For $x, y \in R$, let $[x,y]$ be the commutator $xy - yx$.

The subring S of R is said to be of finite index in R if $(S,+)$ is of finite index in $(R,+)$. An element $x \in R$ is called periodic if there exist distinct positive integers m, n such that $x^m = x^n$; and the ring R is called periodic if each of its elements is periodic.

We will use without explicit mention two well-known facts:

(i) the intersection of finitely many subrings of finite index in R is a subring of finite index in R;

(ii) if R is semiprime and I is an ideal of R, then $R/A(I)$ is semiprime.
We will also need several lemmas.

Lemma 1.1 is a theorem from [6]; Lemma 1.2 appears in [3], and with a different proof in [2]; Lemma 1.3, also given without proof, is all but obvious. Lemma 1.6, which appears to be new, is the key to our proofs of Theorems 1 and 2.

Lemma 1.1. *If R is a ring and S is a subring of finite index in R, then S contains an ideal of R which is of finite index in R.*

Lemma 1.2. *Let R be a ring with the property that for each $x \in R$, there exist $m \in \mathbb{Z}^+$ and $p(t) \in \mathbb{Z}[t]$ such that $x^m = x^{m+1}p(x)$. Then R is periodic.*

Lemma 1.3. *If R is any ring with $N \subseteq T$ and H is any finite set of pairwise orthogonal elements of N, then $\langle H \rangle$ is finite.*

Lemma 1.4. *If R is any ring in which R_2 is finite, then R is of bounded index—that is, $N = R_n$ for some $n \in \mathbb{Z}^+$.*

Proof. Let $M = |R_2|$ and let $x \in N$ such that $x^{2k} = 0$ for $k \geq M+1$; and note that $x^k, x^{k+1}, \ldots, x^{2k-1}$ are all in R_2. Since $k > M$, these elements cannot be distinct; hence there exist $h, j \in \mathbb{Z}^+$ such that $h < j \leq 2k-1$ and $x^h = x^{h+m(j-h)}$ for all $m \in \mathbb{Z}^+$. It follows that $x^h = 0$; hence $y^{2M} = 0$ for all $y \in N$. □

Lemma 1.5. *If R is any FZS-ring, then $N \subseteq T$.*

Proof. Let R be a ring with $N \backslash T \neq \varnothing$, and let $x \in N \backslash T$. Then there exists a smallest $n \in \mathbb{Z}^+$ such that $x^n \in T$, and there exists $k \in \mathbb{Z}^+$ for which $kx^n = 0$. Since $kx^{n-1} \notin T$, $\langle kx^{n-1} \rangle$ is an infinite zero subring of R. □

Lemma 1.6. *If R is any FZS-ring and x is any element of N, then $A(x)$ is of finite index in R. Hence, if S is any finite subset of N, $A(S)$ is of finite index in R.*

Proof. We use induction on the degree of nilpotence. Suppose first that $y^2 = 0$. Define $\Phi : Ry \to R$ by $ry \mapsto [ry, y] = -yry$; and note that $\Phi(Ry)$ is a zero subring of R, hence finite. Thus $\ker\Phi = Ry \cap C_R(y)$ is of finite index in Ry. But it is easily seen that $\ker\Phi$ is a zero ring, hence is finite; consequently, Ry is finite. Now consider $\eta : R \to Ry$ defined by $r \mapsto ry$, and note that $\ker\eta = A_l(y)$ is of finite index in R. Similarly, $A_r(y)$ is of finite index and so is $A(y) = A_l(y) \cap A_r(y)$.

Now assume that $A(x)$ is of finite index for all $x \in N$ with degree of nilpotence less than k, and let $y \in N$ be such that $y^k = 0$. Then $A(y^2)$ is of finite index in R. Define $\Phi : A(y^2)y \to R$ by $sy \mapsto [sy, y]$, $s \in A(y^2)$; and note that both $\Phi(A(y^2)y)$ and $\ker\Phi = A(y^2)y \cap C_R(y)$ are zero rings, so that $A(y^2)y$ is finite. Consider the map $\Psi = A(y^2) \to A(y^2)y$ given by $s \to sy$. Now $\ker\Psi = A(y^2) \cap A_l(y)$ must be of finite index in $A(y^2)$; and since $A(y^2)$ is of finite index in R, $\ker\Psi$ is of finite index in R. It follows that $A_l(y)$ is of finite index in R; and a similar argument shows that $A_r(y)$ is of finite index in R. Therefore $A(y)$ is of finite index in R. □

Lemma 1.7. *Let p be a prime, and let R be a ring such that $pR = \{0\}$.*

(i) *If $a \in R$ and $a^{p^k} = a$, then $a^{p^{mk}} = a$ for all $m \in \mathbb{Z}^+$. Hence if $a, b \in R$ with $a^{p^k} = a$ and $b^{p^j} = b$, there exists $n \in \mathbb{Z}^+$ such that $a^{p^n} = a$ and $b^{p^n} = b$.*

(ii) *If* $a \in R$ *and* $a^{p^k} = a$*, then for each* $s \in \mathbb{Z}$*,* $(sa)^{p^k} = sa$*.*

(iii) *If* R *is reduced and* a *is a periodic element of* R*, then there exists* $n \in \mathbb{Z}^+$ *such that* $a^{p^n} = a$*.*

Proof. (i) is almost obvious, and (ii) follows from the fact that $s^p \equiv s \pmod p$ for all $s \in \mathbb{Z}$. To obtain (iii), note that if R is reduced and a is periodic, then $\langle a\rangle$ is finite, hence a direct sum of finite fields, necessarily of characteristic p. Since $\mathrm{GF}(p^\alpha)$ satisfies the identity $x^{p^\alpha} = x$, the conclusion of (iii) follows by (i). □

2. Proofs of Theorems 1 and 2

Proof of Theorem 1. Suppose R is a counterexample. Note that R is an *FZS*-ring, so $R = T$ by Lemma 1.5. It is easy to see that R contains a maximal finite zero subring S. By Lemma 1.6, $A(S)$ is infinite; and maximality of S forces $A(S)_2 = S$. Thus, by replacing R by $A(S)$, we may assume that R_2 is finite.

By Lemma 1.6, we can construct infinite sequences of pairwise orthogonal elements; and by Lemma 1.4 there is a smallest $M \in \mathbb{Z}^+$ for which R_M contains such sequences. Let $u_1, u_2, \ldots$ be an infinite sequence of pairwise orthogonal elements of R_M. Using Lemma 1.3, we can refine this sequence to obtain an infinite subsequence $v_1, v_2, \ldots$ such that for each $j \geq 2$, $v_j \notin \langle v_1, v_2, \ldots, v_{j-1}\rangle$. Defining V_0 to be $\{v_j^2 \mid j \in \mathbb{Z}^+\}$, we see that $V_0 \subseteq R_{M-1}$ and hence V_0 is finite, so we may assume without loss of generality that there exists a single $s \in R$ such that $v_j^2 = s$ for all $j \in \mathbb{Z}^+$. Take $m \in \mathbb{Z}^+$such that $ms = 0$; and for each $j \in \mathbb{Z}^+$, define $w_j = \sum_{i=1}^{mj} v_i$. Then the w_j form an infinite subset of R_2, contrary to the fact that R_2 is finite. The proof is now complete. □

Proof of Theorem 2. As before, since R is an *FZS*-ring, there is a maximal finite zero subring S; and by Lemma 1.6 $A(S)$ is of finite index in R. By Lemma 1.1, $A(S)$ contains an ideal I of R which is also of finite index in R. Let $C = A(I)$ and let $B = A(C)$. Then $B \supseteq I$, so B is of finite index in R.

Next we show that B is reduced. Let $x \in B$ such that $x^2 = 0$. Then $x \in A(C)$; and since $S \subseteq C$, the maximality of S forces $x \in B \cap C = \{0\}$. Therefore, B is reduced.

The rest of the proof is as in [4]. Since R/B is finite and semiprime, we can write it as $M_1 \oplus \cdots \oplus M_k$, where the M_i are total matrix rings over finite fields. Let $C' = (B+C)/B$ and note that C' is an ideal of R/B and $C' \cong C$. Now C' must be a direct sum of some of the M_i, so $R/B = C' \oplus D'$ where D' is the annihilator of C'. Taking D to be an ideal of R containing B for which $D/B = D'$, and noting that $C'D' = \{0\}$, we have $CD \subseteq B$. But $CD \subseteq C$ as well, so $CD \subseteq B \cap C = \{0\}$ and $D \subseteq A(C) = B$; therefore $D' = \{0\}$ and $C' = R/B$. It follows that $R = B + C$ and hence $R = B \oplus C$; and since $C \cong C'$, C is a direct sum of total matrix rings as required. □

Remark 2.1. In [5], Lanski established the conclusion of Theorem 2 under the apparently stronger hypothesis that N is finite; and his proof uses induction on $|N|$. As we noted in the introduction, it follows from Theorems 1 and 2 that R is an *FZS*-ring if and only if N is finite.

3. A theorem on periodic subrings.

We have noted that if N is infinite, R contains an infinite nil subring. Since periodic elements extend the notion of nilpotent element, it is natural to ask whether there is a periodic analogue—that is, to ask whether a ring with infinitely many periodic elements must have an infinite periodic subring. The answer in general is no, even in the case of commutative rings. The complex field $\mathbb{C}$ is a counterexample, for the set of nonzero periodic elements is the set U of roots of unity, and $u \in U$ implies $2u \notin U$. Moreover, if S is any finite ring, $\mathbb{C} \oplus S$ is also a counterexample; therefore, we restrict our attention to rings R for which $R = T$.

Theorem 3.1. *Let R be a ring with $R = T$. Then a necessary and sufficient condition for R to have an infinite periodic subring is that R contains an infinite set of pairwise-commuting periodic elements.*

Proof. It is known that in any infinite periodic ring R, either N is infinite or the center Z is infinite [4, Theorem 7]. Therefore our condition is necessary.

For sufficiency, suppose that R has infinitely many pairwise-commuting periodic elements. Now R is the direct sum of its p-primary components $R^{(p)}$; and if there exist infinitely many primes $p_1, p_2, p_3, \ldots$ such that $R^{(p_i)}$ contains a nonzero periodic element a_{p_i}, then the direct sum of the rings $\langle a_{p_i} \rangle$ is an infinite periodic subring. Thus, we may assume that only finitely many $R^{(p)}$ contain nonzero periodic elements, so we need only consider the case that $R = R^{(p)}$ for some prime p. Of course we may assume that R is an *FZS*-ring.

Consider the factor ring $\bar{R} = R/P(R)$. Since R is an *FZS*-ring, it follows from Theorem 1 that $P(R)$ is finite, in which case $\bar{R}$ inherits our hypothesis on pairwise-commuting periodic elements. If $\bar{R}$ has an infinite periodic subring $\bar{S}$ and S is its preimage in R, then for all $x \in S$, there exist distinct $m, n \in \mathbb{Z}^+$ such that $x^n - x^m \in P(R) \subseteq N$; hence S is periodic by Lemma 1.2. Thus, we may assume that $R = R^{(p)}$ and that R is a semiprime *FZS*-ring. □

By Theorem 2, write $R = B \oplus C$, where B is reduced and C is finite; and note that B must have an infinite subset H of pairwise-commuting periodic elements. Note also that $pB = \{0\}$, since B is reduced. Let $a, b \in H$, and by Lemma 1.7(i) and (iii) obtain $n \in \mathbb{Z}^+$ such that $a^{p^n} = a$ and $b^{p^n} = b$. It follows at once that $(a-b)^{p^n} = a^{p^n} - b^{p^n} = a - b$ and $(ab)^{p^n} = a^{p^n} b^{p^n} = ab$; and these facts, together with Lemma 1.7(ii) imply that $\langle H \rangle$ is an infinite periodic subring of R.

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