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A SUFFICIENT CONDITION FOR STARLIKENESS OF ORDER α

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Abstract. We obtain a sufficient condition for starlikeness of order α, $|f'(z)-\lambda(f(z)/z)+\lambda-1| < M = M_n(\lambda,\alpha)$, where $\lambda \in [0,1]$, $\alpha \in [0,1)$ and the function $f(z) = z+a_{n+1}z^{n+1}+\cdots$ is analytic in the unit disc U.

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1. Introduction and preliminaries. Denote by U the unit disc of the complex plane

$$U = \{z \in \mathbb{C} : |z| < 1\}. \tag{1.1}$$

Let $\mathcal{H}[U]$ be the space of holomorphic functions in U, and let

$$A_n = \{f \in \mathcal{H}[U],\ f(z) = z+a_{n+1}z^{n+1}+\cdots,\ z \in U\} \tag{1.2}$$

with $A_1 = A$.

Let $\mathcal{H}[a,n]$ denote the class of analytic functions in the unit disc of the form

$$f(z) = a+a_nz^n+a_{n+1}z^{n+1}+\cdots, \quad z \in U. \tag{1.3}$$

Let

$$S^*(\alpha) = \left\{f \in A,\ \operatorname{Re}\frac{zf'(z)}{f(z)} > \alpha,\ z \in U\right\}, \quad 0 \le \alpha < 1, \tag{1.4}$$

be the class of starlike functions of order α in U.

If f and g are analytic in U, then we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there is a function w analytic in U, with $w(0) = 0$, $|w(z)| < 1$, for any $z \in U$, such that $f(z) = g(w(z))$, for $z \in U$.

If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

We use the following subordination result due to Hallenbeck and Ruscheweyh [1, page 71].

Lemma 1.1. *Let h be a convex function with $h(0) = a$, and let $\gamma \in \mathbb{C}^*$ be a complex number with* $\operatorname{Re}\gamma \ge 0$. *If $p \in \mathcal{H}[a,n]$ and*

$$p(z)+\frac{1}{\gamma}zp'(z) \prec h(z), \tag{1.5}$$

then

$$p(z) \prec q(z), \tag{1.6}$$

where

$$q(z)=\frac{\gamma}{nz^{\gamma/n}}\int_0^z h(t)t^{\gamma/n-1}\,dt,\quad q\prec h. \tag{1.7}$$

2. Main results

Theorem 2.1. *Let $\lambda\in[0,1]$, $\alpha\in[0,1)$, and*

$$M=M_n(\lambda,\alpha)=\frac{(1-\alpha)(n+1-\lambda)}{|\lambda-\alpha|+\sqrt{(1-\lambda)^2+(n+1-\lambda)^2}}. \tag{2.1}$$

If $f\in A_n$ satisfies the inequality

$$\left|f'(z)-\lambda\frac{f(z)}{z}+\lambda-1\right|<M_n(\lambda,\alpha), \tag{2.2}$$

with $M_n(\lambda,\alpha)$ given by (2.1), then $f\in S^*(\alpha)$.

Proof. In the case $\lambda=1$, the proof is given in [3]. We suppose that $\lambda\in[0,1)$. If we consider $P(z)=f(z)/z$, then

$$f(z)=zP(z),\qquad f'(z)=P(z)+zP'(z), \tag{2.3}$$

and (2.2) can be written in the following form:

$$\left|P(z)+\frac{zP'(z)}{1-\lambda}-1\right|<\frac{M}{1-\lambda} \tag{2.4}$$

which is equivalent to the differential subordination

$$P(z)+\frac{zP'(z)}{1-\lambda}\prec 1+\frac{M}{1-\lambda}z\equiv h(z), \tag{2.5}$$

and by using Lemma 1.1, we obtain

$$P(z)\prec q(z)=\frac{\gamma}{nz^{\gamma/n}}\int_0^z h(t)t^{\gamma/n-1}\,dt=1+\frac{M}{1-\lambda+n}z. \tag{2.6}$$

Subordination (2.6) is equivalent to

$$|P(z)-1|<\frac{M}{1-\lambda+n}\equiv R. \tag{2.7}$$

After a simple computation, from (2.7) it follows that

$$R<\frac{1-\alpha}{|\lambda-\alpha|}. \tag{2.8}$$

If we put

$$\frac{zf'(z)}{f(z)}=(1-\alpha)p(z)+\alpha, \tag{2.9}$$

then

$$f'(z)=P(z)[(1-\alpha)p(z)+\alpha] \tag{2.10}$$

and (2.2) can be written as

$$\left|P(z)[(1-\alpha)p(z)+\alpha-\lambda]+\lambda-1\right| < M = (1-\lambda+n)R. \tag{2.11}$$

We have to show that (2.11) implies $\operatorname{Re} p(z) > 0$ in U. Suppose that this is false. Since $p(0) = 1$, there exist $z_0 \in U$ and a real ρ, such that $p(z_0) = i\rho$.

Therefore, in order to show that (2.11) implies $\operatorname{Re} p(z) > 0$ in U, it is sufficient to obtain the contradiction from the inequality

$$\left|P(z_0)[(1-\alpha)p(z_0)+\alpha-\lambda]+\lambda-1\right| \geq (1-\lambda+n)R. \tag{2.12}$$

If we let $P(z_0) = P = u+iv$, then

$$\begin{aligned} E &= \left|P[(1-\alpha)i\rho+\alpha-\lambda]+\lambda-1\right|^2 \\ &= |P|^2[(1-\alpha)^2\rho^2+(\alpha-\lambda)^2]-2(1-\lambda)\operatorname{Re}\{P(1-\alpha)i\rho+\alpha-\lambda\}+(1-\lambda)^2 \\ &= (u^2+v^2)(1-\alpha)^2\rho^2+2(1-\lambda)(1-\alpha)v\rho+\left|P(\alpha-\lambda)-(1-\lambda)\right|^2. \end{aligned} \tag{2.13}$$

By using (2.7) and the well-known triangle inequality, one obtains

$$\begin{aligned} \left|P(\alpha-\lambda)-(1-\lambda)\right| &= \left|P(\alpha-\lambda)+\alpha-\lambda-\alpha+\lambda-1+\lambda\right| \\ &= \left|(\alpha-\lambda)(P-1)-(1-\alpha)\right| \\ &\geq 1-\alpha-|\lambda-\alpha|R \end{aligned} \tag{2.14}$$

and we deduce

$$E \geq (u^2+v^2)(1-\alpha)^2\rho^2+2(1-\lambda)(1-\alpha)v\rho+\left[(1-\alpha)-(\lambda-\alpha)R\right]^2. \tag{2.15}$$

If we let

$$\begin{aligned} F(\rho) &= E-M^2 \\ &\geq (u^2+v^2)(1-\alpha)^2\rho^2+2(1-\lambda)(1-\alpha)v\rho \\ &\quad +\left[(1-\alpha)-|\lambda-\alpha|R\right]^2-(1-\lambda+n)^2R^2, \end{aligned} \tag{2.16}$$

then (2.12) holds if $F(\rho) \geq 0$, for any real number ρ.

Because $(u^2+v^2)(1-\alpha)^2 > 0$, the inequality $F(\rho) \geq 0$ holds if the discriminant Δ is negative, that is,

$$\Delta = (1-\alpha)^2\left\{(1-\lambda)^2v^2-(u^2+v^2)\left[\left(1-\alpha-|\lambda-\alpha|R\right)^2-(1-\lambda+n)^2R^2\right]\right\} \leq 0. \tag{2.17}$$

The last inequality is equivalent to

$$\begin{aligned} &v^2\left[(1-\lambda)^2-\left(1-\alpha-|\lambda-\alpha|R\right)^2+(1-\lambda+n)^2R^2\right] \\ &\qquad \leq u^2\left[\left(1-\alpha-|\lambda-\alpha|R\right)^2-(1-\lambda+n)^2R^2\right]. \end{aligned} \tag{2.18}$$

After an easy computation, by using (2.7) we obtain the inequality

$$\frac{v^2}{u^2} \leq \frac{R^2}{1-R^2} \leq \frac{\left(1-\alpha-|\lambda-\alpha|R\right)^2-(1-\lambda+n)^2R^2}{(1-\lambda)^2-\left(1-\alpha-|\lambda-\alpha|R\right)^2+(1-\lambda+n)^2R^2}, \tag{2.19}$$

which is equivalent to $\Delta \leq 0$. Therefore $F(\rho) \leqq 0$, a contradiction of (2.11). It follows that $\operatorname{Re} p(z) > 0$, and

$$\operatorname{Re}\frac{zf'(z)}{f(z)} = \operatorname{Re}(1-\alpha)p(z)+\alpha = (1-\alpha)\operatorname{Re} p(z)+\alpha \geq \alpha \tag{2.20}$$

hence $f \in S^*(\alpha)$. □

If $\lambda = 0$ then

$$M_n(0,\alpha) = \frac{(1-\alpha)(n+1)}{\alpha+\sqrt{(n+1)^2+1}} \tag{2.21}$$

and we obtain the following corollary.

Corollary 2.2. *If $f \in A_n$ and*

$$|f'(z)-1| < \frac{(1-\alpha)(n+1)}{\alpha+\sqrt{(n+1)^2+1}}, \tag{2.22}$$

then $f \in S^*(\alpha)$.

For $\alpha = 0$ this result was obtained in [2].

If $\lambda = 1$,

$$M_n(1,\alpha) = \frac{n(1-\alpha)}{n+1-\alpha}, \tag{2.23}$$

and we obtain the following corollary.

Corollary 2.3 (see [3]). *If $f \in A_n$ and*

$$\left|f'(z)-\frac{f(z)}{z}\right| < \frac{n(1-\alpha)}{n+1-\alpha}, \tag{2.24}$$

then $f \in S^*(\alpha)$.

If $\lambda = \alpha$,

$$M_n(\alpha,\alpha) = \frac{(1-\alpha)(n+1-\alpha)}{\sqrt{(1-\alpha)^2+(1-\alpha+n)^2}}. \tag{2.25}$$

Corollary 2.4. *If $f \in A_n$ and*

$$\left|f'(z)-\alpha\frac{f(z)}{z}+\alpha-1\right| < \frac{(1-\alpha)(n+1-\alpha)}{\sqrt{(1-\alpha)^2+(1-\alpha+n)^2}}, \tag{2.26}$$

then $f \in S^*(\alpha)$.

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