IJMMS 29:11 (2002) 641–650
PII. S0161171202007810
http://ijmms.hindawi.com

THE CLASS OF FUNCTIONS CONVEX IN THE NEGATIVE DIRECTION OF THE IMAGINARY AXIS OF ORDER (α,β)

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Received 22 July 2001

The aim of this paper is to present an analytic characterization of the class of functions convex in the negative direction of the imaginary axis of order (α,β). The method of the proof is based on Julia's lemma.

2000 Mathematics Subject Classification: 30C45.

1. Introduction. In this paper, we are interested in the subclasses of functions convex in the negative (positive) direction of the imaginary axis of order (α,β) denoted by $\mathcal{CV}^-_{\alpha,\beta}$ ($\mathcal{CV}^+_{\alpha,\beta}$).

In [1, 2], the authors defined and studied the class I_α of functions called *angularly accessible in the direction of the imaginary axis*. Applying the method based on the Carathéodory kernel theorem, they showed an analytic characterization of functions in I_α. The same class I_α with applying the Schwarz-Christoffel formulas and a method of approximation by polygons was defined and examined again in [8], where the author used the name *parallel accessible domains (functions) of order* α.

The aim of this paper is to introduce and analytically characterize functions in the class $\mathcal{CV}^-_{\alpha,\beta}$ ($\mathcal{CV}^+_{\alpha,\beta}$). In the case when $\alpha=\beta=1$, the results reduce to the class $\mathcal{CV}^-$ ($\mathcal{CV}^+$) of functions convex in the negative (positive) direction of the imaginary axis. These classes were distinguished as the subclasses of the class of functions convex in the direction of the imaginary axis in [6]. In [4, 5], the author examined the class L_0 of functions called *convex in the direction of the negative real half-axis*. To be precise, an analytic and univalent function f in the unit disk $\mathbb{D}$ belongs to L_0 if and only if for every $w\in f(\mathbb{D})$ the half-line $\{w+t : t\in[0,+\infty)\}$ is contained in $f(\mathbb{D})$. Applying the Carathéodory kernel theorem the author proved, in a quite simple way, an analytic characterization of the class L_0. Since $if\in\mathcal{CV}^+$ and $-if\in\mathcal{CV}^-$ if $f\in L_0$, the same was done for the classes $\mathcal{CV}^+$ and $\mathcal{CV}^-$. In [9], a new proof of analytic formulas for the classes $\mathcal{CV}^-$ ($\mathcal{CV}^+$) based on Julia's lemma were found. The same idea is used in this paper. At the end we notice that the classes $\mathcal{CV}^-_{\alpha,\alpha}$ and $\mathcal{CV}^+_{\alpha,\alpha}$ are proper subclasses of I_α.

2. Preliminaries. Let $\bar{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$, $\mathbb{D}=\{z\in\mathbb{C} : |z|<1\}$ denote the open disk in the plane and let $\mathbb{T}=\partial\mathbb{D}$. Let $\mathcal{S}$ denote the class of all analytic univalent functions in $\mathbb{D}$.

For each $k > 0$, let

$$\mathbb{O}_k = \left\{ z \in \mathbb{D} : \frac{|1-z|^2}{1-|z|^2} < k \right\} \tag{2.1}$$

denote the disk in $\mathbb{D}$ called the *oricycle*. The oricycle $\mathbb{O}_k$ is a disk in $\mathbb{D}$ whose boundary circle $\partial\mathbb{O}_k$ is tangent to $\mathbb{T}$ at $z = 1$.

In the proof of the main theorem, we apply the Julia lemma (see [7]; see also [3, page 56]) recalled below.

Lemma 2.1 (Julia [7]). *Let ω be an analytic function in $\mathbb{D}$ with $|\omega(z)| < 1$ for $z \in \mathbb{D}$. Assume that there exists a sequence (z_n) of points in $\mathbb{D}$ such that*

$$\lim_{n\to\infty} z_n = 1, \qquad \lim_{n\to\infty} \omega(z_n) = 1, \tag{2.2}$$

$$\lim_{n\to\infty} \frac{1-|\omega(z_n)|}{1-|z_n|} = \lambda < \infty. \tag{2.3}$$

Then

$$\frac{|1-\omega(z)|^2}{1-|\omega(z)|^2} \le \lambda \frac{|1-z|^2}{1-|z|^2}, \quad z \in \mathbb{D}, \tag{2.4}$$

and hence, for every $k > 0$,

$$\omega(\mathbb{O}_k) \subset \mathbb{O}_{\lambda k}. \tag{2.5}$$

Remark 2.2. Since

$$\frac{1-|\omega(z)|}{1-|z|} \ge \frac{1-|\omega(0)|}{1+|\omega(0)|}, \quad z \in \mathbb{D}, \tag{2.6}$$

for every function ω analytic in $\mathbb{D}$ with $|\omega(z)| < 1$ for $z \in \mathbb{D}$, the constant λ defined in (2.2) is positive (see [3, page 43]).

3. Convexity in the negative direction of the imaginary axis of order (α, β).

We start with notation. For $w \in \mathbb{C}$ and $\theta \in [0, 2\pi)$, let

$$l[w,\theta] = \{w + te^{i\theta} : t \in [0, +\infty)\}. \tag{3.1}$$

For $A, B \subset \mathbb{C}$ and $w \in \mathbb{C}$, let

$$A \pm B = \{u \pm v \in \mathbb{C} : u \in A \wedge v \in B\}, \qquad A + w = A + \{w\}. \tag{3.2}$$

For fixed $\alpha, \beta \in [0,1]$, let

$$A(\alpha,\beta) = \left\{ z \in \mathbb{C} : -(1-\alpha)\frac{\pi}{2} \le \arg z \le (1-\beta)\frac{\pi}{2} \right\} \cup \{0\}. \tag{3.3}$$

Clearly, $A(0,0) = \{z \in \mathbb{C} : \operatorname{Re} z \ge 0\}$ and $A(1,1) = l[0,0] = [0,\infty)$. Notice that $A(\alpha,\beta)$, when $\alpha \ne 1$ or $\beta \ne 1$, is a closed convex sector with the half-lines $l[0,(1-\beta)\pi/2]$ and $l[0, 2\pi - (1-\alpha)\pi/2]$ as its arms.

Definition 3.1. Fix $\alpha,\beta \in [0,1]$. A domain $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$, is called *convex in the negative direction of the imaginary axis of order* (α,β) if and only if $w + iA(\alpha,\beta)$ is contained in Ω for every $w \in \Omega$. The set of all such domains will be denoted by $\mathcal{X}^-_{\alpha,\beta}$.

Definition 3.2. Let $\mathcal{CV}^-_{\alpha,\beta}$ denote the class of all functions $f \in \mathcal{S}$ such that $f(\mathbb{D}) \in \mathcal{X}^-_{\alpha,\beta}$. Functions in $\mathcal{CV}^-_{\alpha,\beta}$ will be called *convex in the negative direction of the imaginary axis of order* (α,β).

The class $\mathcal{X}^-_{1,1}$ denoted for short by $\mathcal{X}^-$ and the corresponding class $\mathcal{CV}^-_{1,1}$ denoted by $\mathcal{CV}^-$ contain domains and functions called *convex in the negative direction of the imaginary axis*, respectively.

Lemma 3.3. *If* $0 \le \alpha_1 \le \alpha_2 \le 1$, $0 \le \beta_1 \le \beta_2 \le 1$, *and* $\Omega \in \mathcal{X}^-_{\alpha_1,\beta_1}$, *then* $\Omega \in \mathcal{X}^-_{\alpha_2,\beta_2}$.

Since $\mathcal{X}^-_{\alpha,\beta} \subset \mathcal{X}^-$ for all $\alpha,\beta \in [0,1]$, every domain in $\mathcal{X}^-_{\alpha,\beta}$ is simply connected.

It is obvious that, for every $f \in \mathcal{CV}^-_{\alpha,\beta}$, there are some points on $\mathbb{T}$ which "correspond" to infinity lying on the boundary of $f(\mathbb{D})$. In what follows, we will use a kind of the boundary normalization for every $f \in \mathcal{CV}^-_{\alpha,\beta}$ by saying that $z = 1$ corresponds to $\infty \in \partial f(\mathbb{D})$. Since, in general, we cannot extend f on $\mathbb{T}$, in order to be precise, we will apply the notion of prime ends to formulate this normalization. Below we construct a prime end $p_\infty(\Omega)$ for every $\Omega \in \mathcal{X}^-_{\alpha,\beta}$ and next using the prime end theorem we associate $z = 1$ with $p_\infty(\Omega)$.

Since for each $\alpha,\beta \in [0,1]$, $\mathcal{CV}^-_{\alpha,\beta} \subset \mathcal{CV}^-$, we can construct for every domain Ω in $\mathcal{X}^-_{\alpha,\beta}$, a prime end $p_\infty(\Omega)$ in this way like in [9].

Construction of a prime end for the domain convex in the negative direction of the imaginary axis. When $\alpha = \beta = 1$ the detailed construction was presented in [9]. The same construction is valid for $\alpha \neq 1$ or $\beta \neq 1$ since $\mathcal{CV}^-_{\alpha,\beta} \subset \mathcal{CV}^-$ for all $\alpha,\beta \in [0,1]$. But in what follows we need some notations used in the construction, so we recall it again.

Let $\Omega \in \mathcal{X}^-_{\alpha,\beta}$.

(1) Assume first that Ω is neither a vertical strip nor a half-plane with the boundary straight line parallel to the imaginary axis. Then there exists $w_0 \in \partial\Omega$ such that $(w_0 + iA(\alpha,\beta)) \setminus \{w_0\}$ lies in Ω. Hence $(l[w_0,\pi/2] \setminus \{w_0\}) \subset \Omega$. For each $t \in (0,\infty)$, we denote $C(t) = \{w \in \mathbb{C} : |w - w_0| = t\}$. It is clear that $\Omega \cap C(t) \neq \emptyset$ for every $t \in (0,\infty)$. By [10, Proposition 2.13, page 28], for each $t \in (0,\infty)$ there are countably many crosscuts $C_k(t) \subset C(t)$, $k \in \mathbb{N}$, of Ω each of which is an arc of the circle $C(t)$. By $\Omega_0(t) \subset \Omega$ we denote the component of $\Omega \setminus C(t)$ containing the half-line $l[w_0 + it, \pi/2] \setminus \{w_0 + it\}$ and by $Q(t) \in \bigcup_{k\in\mathbb{N}} C_k(t)$ we denote the crosscut containing the point $w_0 + it$. So $Q(t) \subset \partial\Omega_0(t)$. Let now (t_n) be a strictly increasing sequence of points in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \infty$ and let $(Q(t_n))$ be the corresponding sequence of crosscuts of Ω. It is easy to observe that

- (i) $\overline{Q(t_n)} \cap \overline{Q(t_{n+1})} = \emptyset$ for every $n \in \mathbb{N}$;
- (ii) $\Omega_0(t_{n+1}) \subset \Omega_0(t_n)$ for every $n \in \mathbb{N}$;
- (iii) $\operatorname{diam}^{\#} Q(t_n) \to 0$ as $n \to \infty$, where $\operatorname{diam}^{\#} B$ means the spherical diameter of the set $B \subset \mathbb{C}$.

Therefore $(C_n) = (Q(t_n))$ forms a null chain of Ω (see [10, page 29]). Notice also that the null chain (C_n) is independent of the choice of the sequence (t_n).

The equivalence class of the null chain (C_n) defines the prime end denoted by $p_\infty(\Omega)$. We can also show that infinity is a unique principal point of the prime end $p_\infty(\Omega)$.

(2) (a) Let Ω be a vertical strip of width $d > 0$. Clearly, this is possible only when $\alpha = \beta = 1$. Let $w_0 \in \partial\Omega$. For each $t \in (d,\infty)$, set $C(t) = \{w \in \mathbb{C} : |w - w_0| = t\}$. It is clear that $\Omega \cap C(t) \neq \emptyset$ for every $t \in (d,\infty)$. Observe that $\Omega(t)$ is a sum of two disjoint circular arcs, denoted by $Q^+(t)$ and $Q^-(t)$. Let $Q^+(t)$ be the circular arc which lies above $Q^-(t)$. Precisely, $Q^+(t)$ cuts the boundary straight lines of Ω at two points: $w_1(t)$ and $w_2(t)$, and together with half-lines $l[w_1(t),\pi/2]$ and $l[w_2(t),\pi/2]$ is a boundary of a domain denoted by $\Omega^+(t)$. Moreover, $\Omega^+(t) \subset \Omega$ and $\Omega^+(t) \cap \operatorname{Int} C(t) = \emptyset$.

Let now (t_n) be a strictly increasing sequence of points in (d,∞) such that $\lim_{n\to\infty} t_n = \infty$, and let $(Q^+(t_n))$ be the corresponding sequence of crosscuts of Ω. It is easy to observe that the conditions (i)–(iii) listed in part (1) are fulfilled. Therefore $(C_n^+) = (Q^+(t_n))$ forms a null chain of Ω. The null chain (C_n^+) is independent of the choice of the sequence (t_n).

The equivalence class of the null chain (C_n^+) defines the prime end denoted by $p_\infty^+(\Omega)$. We can also say that infinity is a unique principal point of the prime end $p_\infty^+(\Omega)$.

In a similar way the sequence $(Q^-(t_n))$ is a null chain which represents the second prime end $p_\infty^-(\Omega)$, different from $p_\infty^+(\Omega)$.

For the next considerations, the prime end $p_\infty^+(\Omega)$ will be denoted by $p_\infty(\Omega)$.

(b) Let now Ω be a half-plane with the boundary straight line parallel to the imaginary axis. Let $w_0 \in \partial\Omega$, and for each $t \in (0,\infty)$, let $C(t) = \{w \in \mathbb{C} : |w - w_0| = t\}$. It is clear that $Q(t) = \Omega \cap C(t)$ is a halfcircle for every $t > 0$. Repeating considerations similar to those above we see that the sequence $(C_n) = (Q(t_n))$, for an arbitrary strictly increasing sequence (t_n) of points in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \infty$, forms a null chain of Ω which represents a prime end denoted by $p_\infty(\Omega)$.

In this way, we construct for every $\Omega \in \mathscr{X}^-_{\alpha,\beta}$, in a unique way, a prime end $p_\infty(\Omega)$. We can also show that infinity is a unique principal point of the prime end $p_\infty(\Omega)$.

Therefore, the following proposition follows.

Proposition 3.4. *For every $\Omega \in \mathscr{X}^-_{\alpha,\beta}$, $\alpha,\beta \in [0,1]$, the prime end $p_\infty(\Omega)$ is of the first or of the second kind.*

Let $f \in \mathscr{CV}^-_{\alpha,\beta}$ and $\Omega = f(\mathbb{D})$. By the prime end theorem there exists a bijective mapping $\hat{f}$ of the unit circle $\mathbb{T}$ onto the set of all prime ends of Ω (see [10, page 30]). Hence there is a unique $\zeta_\infty \in \mathbb{T}$ such that $p_\infty(\Omega) = \hat{f}(\zeta_\infty)$. We can also show that infinity is a unique principal point of the prime end $p_\infty(\Omega)$.

If now $f \in \mathscr{CV}^-_{\alpha,\beta}$, then we can write $p_\infty(\Omega) = \hat{f}(\zeta_\infty)$ for unique $\zeta_\infty \in \mathbb{T}$.

4. An analytic characterization of the class of function convex in the negative direction of the imaginary axis of order (α,β). In the proof of the main theorem, which analytically characterizes the class $\mathscr{CV}^-_{\alpha,\beta}$, we need the following lemma which was proved in [9] in an easy way.

Lemma 4.1. *Every sequence (a_n) of positive numbers with*

$$\lim_{n\to\infty}(a_1 a_2 \cdots a_n) = 0 \tag{4.1}$$

has a convergent subsequence (a_{n_k}) *and*

$$0 \le \lim_{k\to\infty} a_{n_k} = a \le 1. \tag{4.2}$$

Now we prove the theorem which says that every function $f \in \mathcal{CV}^-_{\alpha,\beta}$, with $p_\infty(f(\mathbb{D})) = \hat{f}(1)$, preserves convexity in the negative direction of the imaginary axis of order (α,β) on every oricycle $\mathbb{O}_k$.

Theorem 4.2. *Let $\alpha,\beta \in [0,1]$ and $f \in \mathcal{S}$. Then $f \in \mathcal{CV}^-_{\alpha,\beta}$ and $p_\infty(f(\mathbb{D})) = \hat{f}(1)$, if and only if $f(\mathbb{O}_k) \in \mathfrak{X}^-_{\alpha,\beta}$ for every $k > 0$.*

Proof. (1) Assume that $f \in \mathcal{CV}^-_{\alpha,\beta}$ and $\zeta_\infty = 1$ corresponds to the prime end $p_\infty(f(\mathbb{D}))$. For each $u \in A(\alpha,\beta)$, let

$$\omega_u(z) = f^{-1}(f(z)+iu), \quad z \in \mathbb{D}. \tag{4.3}$$

Since $f(\mathbb{D}) \in \mathfrak{X}^-_{\alpha,\beta}$, $f(z)+iu \in f(\mathbb{D})$ for every $u \in A(\alpha,\beta)$ and $z \in \mathbb{D}$. Hence, from the univalence of f, it follows that ω_u is well defined.

Fix $u \in A(\alpha,\beta)$ and let $\Omega \in \mathfrak{X}^-_{\alpha,\beta}$.

We select two points: $w_0 \in \partial\Omega$ and $w_1 \in \Omega$, in the following way. If Ω is not a vertical strip or a half-plane with the boundary straight line parallel to the imaginary axis, then there exists $w_0 \in \partial\Omega$ such that $(w_0 + iA(\alpha,\beta)) \setminus \{w_0\}$ lies in Ω. Since $(w_0+iu) \in (w_0 + iA(\alpha,\beta))$, the half-line l starting from w_0 and going through u lies in $w_0 + iA(\alpha,\beta)$. Consequently, $(l \setminus \{w_0\}) \subset \Omega$. Fix $w_1 \in l \setminus \{w_0\}$.

In the case when Ω is a vertical strip or a half-plane with the boundary straight line parallel to the imaginary axis, let $w_1 \in \Omega$ be arbitrary and $w_0 \in \partial\Omega$ be such that $\operatorname{Im} w_1 = \operatorname{Im} w_0$.

Assume now that, for $\Omega = f(\mathbb{D})$, the points w_0 and w_1 are chosen as above. Consider the sequence $(w_n) = (w_1 + i(n-1)u)$ of points in $l \setminus \{w_0\}$ and the corresponding sequence $(z_n) = (f^{-1}(w_n))$ of points in $\mathbb{D}$.

With a notation as in the construction of a prime, end let $C(t_n) = \{w \in \mathbb{C} : |w - w_0| = |w_n - w_0|\}$, where $t_n = |w_n - w_0|$, and let $Q(t_n) \subset C(t_n)$, for $n \in \mathbb{N}$, denote the crosscut of $f(\mathbb{D})$ containing w_n. From the method of choosing w_0 and w_1, we see that the conditions (1)–(3) are satisfied and $(Q(t_n))$ is a null chain representing the prime end $p_\infty(f(\mathbb{D}))$. By the prime end theorem $(f^{-1}(Q(t_n)))$ is a null-chain in $\mathbb{D}$ that separates the origin from $\zeta_\infty = 1$ for large n. Since $z_n = f^{-1}(w_n) \in f^{-1}(Q(t_n))$ and $\operatorname{diam} f^{-1}(Q(t_n)) \to 0$ for $n \to \infty$, we conclude that $\lim_{n\to\infty} z_n = 1$. Observe that

$$\omega_u(z_n) = f^{-1}(w_n + iu) = z_{n+1}. \tag{4.4}$$

Let now

$$a_n = \frac{1 - |\omega_u(z_n)|}{1 - |z_n|}, \quad n \in \mathbb{N}. \tag{4.5}$$

Hence

$$a_n = \frac{1 - |\omega_u(z_n)|}{1 - |z_n|} = \frac{1 - |z_{n+1}|}{1 - |z_n|}, \tag{4.6}$$

for all $n \in \mathbb{N}$. Consequently,

$$\begin{aligned}\lim_{n\to\infty}(a_1a_2\cdots a_n) &= \lim_{n\to\infty}\left(\frac{1-|z_2|}{1-|z_1|}\frac{1-|z_3|}{1-|z_2|}\cdots\frac{1-|z_n|}{1-|z_{n-1}|}\frac{1-|z_{n+1}|}{1-|z_n|}\right)\\ &= \lim_{n\to\infty}\frac{1-|z_{n+1}|}{1-|z_1|}=0.\end{aligned}\tag{4.7}$$

By Lemma 4.1, there exists a convergent subsequence (a_{n_k}) of the sequence (a_n) such that

$$0 \le \lim_{k\to\infty} a_{n_k} = \lambda(u) \le 1. \tag{4.8}$$

Hence we conclude that, for each $u \in A(\alpha,\beta)$, there exists a convergent subsequence (z_{n_k}) of the sequence (z_n) such that

$$\lim_{k\to\infty}\frac{1-|\omega_u(z_{n_k})|}{1-|z_{n_k}|} = \lambda(u) \le 1. \tag{4.9}$$

In view of Remark 2.2, $\lambda(u) > 0$ for every $u \in A(\alpha,\beta)$. By this way, ω_u satisfies the assumptions of the Julia lemma with $\lambda(u) \in (0,1]$. Hence

$$\omega_u(\mathbb{O}_k) \subset \mathbb{O}_{\lambda(u)k} \subset \mathbb{O}_k \tag{4.10}$$

for every $u \in A(\alpha,\beta)$ and $k > 0$. This yields $f^{-1}(f(\mathbb{O}_k)+iu) \subset \mathbb{O}_k$, so $f(\mathbb{O}_k)+iu \subset f(\mathbb{O}_k)$ for every $u \in A(\alpha,\beta)$. Therefore $f(\mathbb{O}_k) \in \mathfrak{X}^-_{\alpha,\beta}$ for every $k > 0$.

(2) Now assume that $f(\mathbb{O}_k) \in \mathfrak{X}^-_{\alpha,\beta}$ for every $k > 0$. Since $\infty \in \partial f(\mathbb{O}_k)$ for every $k > 0$ and

$$f(\mathbb{D}) = \bigcup_{k>0} f(\mathbb{O}_k), \tag{4.11}$$

$\infty \in \partial f(\mathbb{D})$ and $f(\mathbb{D}) \in \mathfrak{X}^-_{\alpha,\beta}$. Observe also that there exists a prime end $p_\infty(f(\mathbb{D}))$ which corresponds to some point $\zeta_\infty \in \mathbb{T}$. We need to show that $\zeta_\infty = 1$.

To this end, let $k > 0$ be fixed and suppose that $\zeta_\infty \ne 1$.

Let $(Q(t_n))$ be an arbitrary sequence of crosscuts of $f(\mathbb{D})$ which represents the prime end $p_\infty(f(\mathbb{D}))$ corresponding in a unique way to a point $\zeta_\infty \in \mathbb{T}$, that is, $(Q(t_n))$ is a null-chain of $f(\mathbb{D})$. By the prime end theorem $(f^{-1}(Q(t_n)))$ is a null-chain that separates in $\mathbb{D}$ the origin and ζ_∞ for large n. Since $\zeta_\infty \ne 1$ and $\operatorname{diam} f^{-1}(Q(t_n)) \to 0$ for $n \to \infty$ we see that

$$f^{-1}(Q(t_n)) \cap \mathbb{O}_k = \varnothing, \tag{4.12}$$

for large n.

On the other hand, $f(\mathbb{O}_k) \in \mathfrak{X}^-_{\alpha,\beta}$, which implies that $Q(t_n) \cap f(\mathbb{O}_k) \ne \varnothing$ for large $n \in \mathbb{N}$. This contradicts (4.12) and shows that $\zeta_\infty = 1$ and $p_\infty(f(\mathbb{D})) = \hat{f}(1)$. The proof of the theorem is finished. □

Using Theorem 4.2 we find an analytic characterization of functions in the class $\mathscr{CV}^-_{\alpha,\beta}$.

Theorem 4.3. *Let $\alpha,\beta\in[0,1]$. If $f\in\mathscr{CV}^{-}_{\alpha,\beta}$ and $p_\infty(f(\mathbb{D}))=\hat{f}(1)$, then*

$$-\beta\frac{\pi}{2}\le\arg\{-i(1-z)^2f'(z)\}\le\alpha\frac{\pi}{2},\quad z\in\mathbb{D}. \tag{4.13}$$

Proof. Let $\Omega=f(\mathbb{D})$. The case $\alpha=\beta=1$ is well known and can be found in [5, 6, 9]. Assume that $\alpha\ne1$ or $\beta\ne1$. This means that $A(\alpha,\beta)$ is a closed convex sector which does not reduce to the half-line $l[0,0]$. Now we prove that (4.13) is true for all points on $\gamma_k=\partial\mathbb{O}_k\setminus\{1\}$ for every $k>0$. We use the following parametrization of γ_k:

$$\gamma_k: z=z(\theta)=\frac{1+ke^{i\theta}}{1+k},\quad \theta\in(0,2\pi). \tag{4.14}$$

Let $\Gamma_k=f(\gamma_k)$, since γ_k is positively oriented, so is Γ_k. For each $z\in\gamma_k$ we denote by $\tau(z)$ the tangent vector to Γ_k at $w=f(z)$, that is,

$$\tau(z)=z'(\theta)f'(z(\theta)), \tag{4.15}$$

where $z=z(\theta)$ is given by (4.14). Since

$$\begin{aligned}(1-z(\theta))^2&=\frac{k^2}{(1+k)^2}\left(1-e^{i\theta}\right)^2=\frac{4k\sin^2(\theta/2)}{k+1}z'(\theta)i\\&=2\operatorname{Re}\{1-z(\theta)\}z'(\theta)i,\quad\theta\in(0,2\pi),\\ \tau(z)&=-\frac{i(1-z)^2f'(z)}{2\operatorname{Re}\{1-z\}},\quad z\in\gamma_k.\end{aligned} \tag{4.16}$$

Let V denote the closed convex sector with vertex at w and with the half-lines $l[w,\alpha\pi/2]$ and $l[w,2\pi-\beta\pi/2]$ as its arms. This means that $(w+iA(\alpha,\beta))\cup V$ forms a closed half-plane containing the half-line $l[w,\alpha\pi/2]$.

Fix $k>0$. By Theorem 4.2, $f(\mathbb{O}_k)\in\mathscr{L}^{-}_{\alpha,\beta}$ for every $k>0$. Therefore by an easy observation we see that $w+iA(\alpha,\beta)\subset\overline{f(\mathbb{O}_k)}$. Hence it follows that the tangent line to Γ_k at w cannot intersect the interior of the sector $w+iA(\alpha,\beta)$. This implies that $\tau(z)$ lies in V. Consequently, in view of (4.16), we have

$$-\beta\frac{\pi}{2}\le\arg\{\tau(z)\}=\arg\{-i(1-z)^2f'(z)\}\le\alpha\frac{\pi}{2} \tag{4.17}$$

for $z\in\gamma_k$. As k was arbitrary, this is true in $\mathbb{D}$. □

Now we prove the converse theorem.

Theorem 4.4. *Let $\alpha,\beta\in[0,1]$. If $f\in\mathscr{A}$ and (4.13) is true, then $f\in\mathscr{CV}^{-}_{\alpha,\beta}$ and $p_\infty(f(\mathbb{D}))=\hat{f}(1)$.*

Proof. (1) Suppose that there exists $z_0\in\mathbb{D}$ such that the equality in the left-hand side of (4.13) holds. Then it holds in the whole disk $\mathbb{D}$, that is, there exists a positive real number a such that

$$-i(1-z)^2f'(z)\equiv ae^{-i\beta\pi/2},\quad z\in\mathbb{D}. \tag{4.18}$$

This is satisfied only for

$$f(z) = b + \frac{aie^{-i\beta\pi/2}}{1-z}, \quad z \in \mathbb{D}, \tag{4.19}$$

where $b \in \mathbb{C}$.

In a similar way, if the equality in the right-hand side of (4.13) holds for some $z_0 \in \mathbb{D}$, then it holds only for

$$f(z) = b + \frac{aie^{i\alpha\pi/2}}{1-z}, \quad z \in \mathbb{D}, \tag{4.20}$$

where $b \in \mathbb{C}$.

Particularly, if $\alpha = \beta = 0$, then (4.13) is true only for

$$f(z) = b + \frac{ai}{1-z}, \quad z \in \mathbb{D}, \tag{4.21}$$

where $b \in \mathbb{C}$ and $a \in \mathbb{R} \setminus \{0\}$.

Functions (4.19) and (4.20) map $\mathbb{D}$ onto half-planes and a simple geometric viewing shows that they are elements of $\mathscr{CV}^-_{\alpha,\beta}$ with $p_\infty(f(\mathbb{D})) = \hat{f}(1)$.

(2) Suppose that in (4.13) strong inequalities holds. Since $f \in \mathscr{CV}^-$, f is univalent in $\mathbb{D}$ (see [5, 4, 6]). We show that $f(\mathbb{D}) \in \mathscr{X}^-_{\alpha,\beta}$.

Suppose, on the contrary, that $f(\mathbb{D}) \notin \mathscr{X}^-_{\alpha,\beta}$. By Theorem 4.2, there exists $k > 0$ such that $f(\mathbb{O}_k) \notin \mathscr{X}^-_{\alpha,\beta}$. This means that $(w_0 + iA(\alpha,\beta)) \setminus \{w_0\}$ is not contained in $f(\mathbb{O}_k)$ for some $w_0 \in f(\mathbb{O}_k)$.

Suppose that

$$\Gamma_k \cap l\left[w_0, \pi - \beta\frac{\pi}{2}\right] \neq \varnothing. \tag{4.22}$$

Thus there exists $w_1 \in \Gamma_k \cap l[w_0, \pi - \beta\pi/2]$, $w_1 \neq w_0$, such that the segment $[w_0, w_1)$ lies in $f(\mathbb{O}_k)$. Let $\tau(z_1)$ be the tangent vector to Γ_k at $w_1 = f(z_1)$, where $z_1 \in \gamma_k$. Let V denote the closed convex sector with vertex at w_1 and with the half-lines $l[w_1, \alpha\pi/2]$ and $l[w_1, 2\pi - \beta\pi/2]$ as its arms. This means that $(w_1 + iA(\alpha,\beta)) \cup V$ is a closed half-plane containing the half-line $l[w_1, \alpha\pi/2]$. Let H be the complementary closed half-plane. Hence $\tau(z_1)$ lies in H which means that

$$\arg\{\tau(z_1)\} \in \left[-\pi, -\beta\frac{\pi}{2}\right] \cup \left[\pi - \beta\frac{\pi}{2}, \pi\right], \tag{4.23}$$

contrary to (4.13).

In a similar way we obtain a contradiction assuming that

$$\Gamma_k \cap l\left[w_0, \alpha\frac{\pi}{2}\right] \neq \varnothing. \tag{4.24}$$

This ends the proof. □

Remark 4.5. For $\alpha = \beta = 1$, Theorems 4.3 and 4.4 show the well-known analytic characterization of the class $\mathscr{CV}^-$.

The following theorems are immediate consequences of Theorems 4.3 and 4.4 by applying them to the function $f(z) = g(e^{-i\mu}z)$, $z \in \mathbb{D}$, where $g \in \mathscr{CV}^-_{\alpha,\beta}$ and $p_\infty(g(\mathbb{D})) = \hat{g}(1)$.

Theorem 4.6. *Let $\alpha,\beta \in [0,1]$. If $f \in \mathscr{CV}^-_{\alpha,\beta}$ and $p_\infty(f(\mathbb{D})) = \hat{f}(e^{i\mu})$, $\mu \in \mathbb{R}$, then*

$$-\beta\frac{\pi}{2} \le \arg\{-ie^{i\mu}(1-e^{-i\mu}z)^2 f'(z)\} \le \alpha\frac{\pi}{2}, \quad z \in \mathbb{D}. \tag{4.25}$$

Theorem 4.7. *Let $\alpha,\beta \in [0,1]$. If $f \in \mathscr{A}$ and (4.25) is true for $\mu \in \mathbb{R}$, then $f \in \mathscr{CV}^-_{\alpha,\beta}$ and $p_\infty(f(\mathbb{D})) = \hat{f}(e^{i\mu})$.*

5. Convexity in the positive direction of the imaginary axis of order (α,β). The results presented in Section 4 can be applied at once to the functions called *convex in the positive direction of the imaginary axis of order* (α,β).

Definition 5.1. Fix $\alpha,\beta \in [0,1]$. A domain $\Omega \subset \mathbb{C}$, $\Omega \ne \mathbb{C}$, will be called *convex in the positive direction of the imaginary axis of order* (α,β) if and only if the sector $w - iA(\alpha,\beta)$ is contained in Ω for every $w \in \Omega$. The set of all such domains will be denoted by $\mathscr{X}^+_{\alpha,\beta}$.

Definition 5.2. Let $\mathscr{CV}^+_{\alpha,\beta}$ denote the class of all functions $f \in \mathscr{S}$ such that $f(\mathbb{D}) \in \mathscr{X}^+_{\alpha,\beta}$. Functions in the class $\mathscr{CV}^+_{\alpha,\beta}$ will be called *convex in the positive direction of the imaginary axis of order* (α,β).

Since $f \in \mathscr{CV}^+_{\alpha,\beta}$ if and only if $-f \in \mathscr{CV}^-_{\alpha,\beta}$ we have the following theorems.

Theorem 5.3. *Let $\alpha,\beta \in [0,1]$ and let $f \in \mathscr{S}$. Then $f \in \mathscr{CV}^+_{\alpha,\beta}$ and $p_\infty(f(\mathbb{D})) = \hat{f}(1)$, if and only if $f(\mathbb{O}_k) \in \mathscr{X}^+_{\alpha,\beta}$ for every $k > 0$.*

Theorem 5.4. *Let $\alpha,\beta \in [0,1]$. If $f \in \mathscr{CV}^+_{\alpha,\beta}$ and $p_\infty(f(\mathbb{D})) = \hat{f}(e^{i\mu})$, $\mu \in \mathbb{R}$, then*

$$-\beta\frac{\pi}{2} \le \arg\left\{ie^{i\mu}(1-e^{-i\mu}z)^2 f'(z)\right\} \le \alpha\frac{\pi}{2}, \quad z \in \mathbb{D}. \tag{5.1}$$

Theorem 5.5. *Let $\alpha,\beta \in [0,1]$. If $f \in \mathscr{A}$ and (5.1) is true for $\mu \in \mathbb{R}$, then $f \in \mathscr{CV}^+_{\alpha,\beta}$ and $p_\infty(f(\mathbb{D})) = \hat{f}(e^{i\mu})$.*

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