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ON A CLASS OF EVEN-DIMENSIONAL MANIFOLDS STRUCTURED BY AN AFFINE CONNECTION

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We deal with a $2m$-dimensional Riemannian manifold (M,g) structured by an affine connection and a vector field $\mathcal{T}$, defining a $\mathcal{T}$-parallel connection. It is proved that $\mathcal{T}$ is both a torse forming vector field and an exterior concurrent vector field. Properties of the curvature 2-forms are established. It is shown that M is endowed with a conformal symplectic structure Ω and $\mathcal{T}$ defines a relative conformal transformation of Ω.

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1. Introduction. In [5], a class of odd-dimensional manifolds endowed with a $\mathcal{T}$-parallel connection was investigated.

In the present paper, we consider a $2m$-dimensional Riemannian manifold (M,g), structured by an affine connection defined by the torsion 2-forms S^A, $A \in \{1,2,\dots,2m\}$. If $\{e_A\}$ and $\{\omega^A\}$ are a vector and a covector basis, respectively, and $\mathcal{T}(T^A)$ a vector field (called the *structure* vector field of M), we assume that $\mathcal{T}$ defines a $\mathcal{T}$-parallel connection, in the sense of [9] (see also [2, 4]), that is, the connection forms associated with $\{e_A\}$ and $\{\omega^A\}$ satisfy

$$\theta_B^A = \langle \mathcal{T}, e_B \wedge e_A \rangle = T^B \omega^A - T^A \omega^B, \tag{1.1}$$

where $\wedge$ means the wedge product of vector fields, which implies $\nabla_{\mathcal{T}} e_A = 0$.

Next, we assume that the torsion forms S^A are *exterior recurrent* (abbreviated ER) [1] with $\alpha = \mathcal{T}^\flat$ as recurrence form, that is, $dS^A = \alpha \wedge S^A$.

Assuming that T^A are also ER with a certain Pfaffian u as recurrence form, that is, $dT^A = T^A u$, and denoting $2t = \|\mathcal{T}\|^2$, we have

$$\nabla \mathcal{T} = 2t\,dp + (u-\alpha) \otimes \mathcal{T}, \tag{1.2}$$

where dp is the soldering form of M [3], which says that $\mathcal{T}$ is a *torse forming* vector field [8, 11, 12].

We derive

$$\nabla^2 \mathcal{T} = 2t(u+\alpha) \wedge dp, \tag{1.3}$$

that is, $\mathcal{T}$ is an *exterior concurrent* vector field [10] (see also [4]).

Setting $S = S^1 \wedge S^2 \wedge \cdots \wedge S^{2m}$, we find that the $4m$-form S associated with M is ER with $4m\alpha$ as recurrence form.

It is shown that the curvature 2-forms Θ^A_B are ER having the closed 1-form $2(u+\alpha)$ as recurrence form. We agree to define such a manifold as an *exterior recurrent curvature 2-form* manifold.

Finally, assuming that M carries an *almost symplectic form* Ω, that is, a nondegenerate differential 2-form, we prove that Ω is a *conformal* symplectic form.

It is shown that $\mathcal{T}$ defines a *relative conformal transformation* of the conformal symplectic form Ω (see [5]).

The above results are stated in Theorem 3.1.

2. Preliminaries. Let (M,g) be a $2m$-dimensional oriented Riemannian manifold structured by an affine differential operator ∇.

Let $\Gamma(TM)$ be the set of sections of the tangent bundle and $\flat : TM \to T^*M$ and $\sharp : T^*M \to TM$ the classical musical isomorphisms defined by g (i.e., $\flat$ is the index lowering operator and $\sharp$ is the index raising operator).

Following [7], we denote by

$$A^q(M,TM) = \Gamma \operatorname{Hom}(\wedge^q TM, TM) \tag{2.1}$$

the set of vector-valued q-forms ($q \le \dim M$) and we write for the affine operator ∇

$$d^\nabla : A^q(M,TM) \longrightarrow A^{q+1}(M,TM). \tag{2.2}$$

If $dp \in A^1(M,TM)$ is the canonical vector-valued 1-form of M, then as an extension of the Levi-Civita operator and by [3], we agree to call dp the *soldering form* of M.

Let the unit vector fields $\{e_A\}$ be an orthonormal vector basis and $\{\omega^A\}$ its corresponding cobasis on M, $A = 1,\dots,2m$. Then, if θ^A_B, S^A, and Θ^A_B denote the connection forms, the torsion 2-forms and the curvature 2-forms, respectively, Cartan's structure equations are expressed by

$$\nabla e_A = \theta^A_B \otimes e_B, \tag{2.3}$$

$$d\omega^A = \omega^B \wedge \theta^A_B + S^A, \tag{2.4}$$

$$d\theta^A_B = \theta^C_B \wedge \theta^A_C + \Theta^A_B. \tag{2.5}$$

We recall the following definitions (cf. [4]).

A vector field $\mathcal{T}$ is said to be a *torse forming* vector field [12] if it satisfies

$$\nabla \mathcal{T} = f\mathcal{T} + v \otimes \mathcal{T}, \quad f \in C^\infty M,\ v \in \wedge^1 M. \tag{2.6}$$

Also, the vector field $\mathcal{T}$ is called exterior concurrent [10] if

$$\nabla^2 \mathcal{T} = \pi \wedge dp, \quad \pi \in \wedge^1 M. \tag{2.7}$$

If $Z, Z' \in \Gamma(TM)$, we also have the following formula:

$$d\omega(Z,Z') = \mathcal{L}_{Z'}\omega(Z) - \mathcal{L}_Z\omega(Z') + \omega([Z,Z']), \tag{2.8}$$

where $\mathcal{L}$ is the Lie derivative.

Since $dp = \omega^A \wedge e_A$, then it follows that

$$d^{\nabla}(dp) = S^A \otimes e_A. \tag{2.9}$$

3. Manifolds with affine connection. In the present paper, we assume first that the $2m$-dimensional Riemannian manifold (M,g) carries a structure vector field $\mathcal{T}(T^A)$ which defines a $\mathcal{T}$-parallel connection, in the sense of [9] (see also [2, 4]). Such a connection is expressed by

$$\theta_B^A = \langle \mathcal{T}, e_B \wedge e_A \rangle = T^B \omega^A - T^A \omega^B. \tag{3.1}$$

Since we quickly find from (3.1) that

$$\nabla_{\mathcal{T}} e_A = 0, \tag{3.2}$$

this agrees with the definition of $\mathcal{T}$-parallel connection.

Setting $2t = \|T\|^2$, we derive

$$\nabla \mathcal{T} = 2t\,dp - \alpha \otimes \mathcal{T} + \sum_A dT^A \otimes e_A, \tag{3.3}$$

where $\alpha = \mathcal{T}^\flat$ is the dual 1-form of $\mathcal{T}$. Also, we find by (3.1) and (2.4) that

$$d\omega^A = \alpha \wedge \omega^A + S^A. \tag{3.4}$$

Second, we assume that the torsion forms S^A are exterior recurrent [1] having α as recurrence form, that is,

$$dS^A = \alpha \wedge S^A, \tag{3.5}$$

and T^A are ER with the Pfaffian u as recurrence form, that is,

$$dT^A = T^A u. \tag{3.6}$$

We obtain $d\alpha = 0$, that is, $\alpha^\sharp = \mathcal{T}$ is a closed vector field.

Under these conditions, it follows from (3.3) and (3.6) that

$$\nabla \mathcal{T} = 2t\,dp + (u - \alpha) \otimes \mathcal{T}; \tag{3.7}$$

this proves that $\mathcal{T}$ is a torse forming vector field [4, 8, 11, 12]. Since the operator ∇ acts inductively and clearly by (3.6), then

$$dt = 2tu, \tag{3.8}$$

we infer

$$d^{\nabla}(\nabla \mathcal{T}) = \nabla^2 \mathcal{T} = 2t(u + \alpha) \wedge dp. \tag{3.9}$$

This means that the vector field $\mathcal{T}$ is an exterior concurrent vector field [6, 10].

By [6], (3.9) implies that

$$\mathcal{R}(\mathcal{T},Z) = -(2m-1)2tg(\mathcal{T},Z), \quad Z \in \Gamma(TM), \tag{3.10}$$

where $\mathcal{R}$ denotes the Ricci tensor field on M.

By (3.9) and by standard calculation, we derive

$$\nabla^4 \mathcal{T} = 0 \tag{3.11}$$

and therefore we may say that the vector field $\mathcal{T}$ is an element of

$$\Gamma \operatorname{Hom}(\wedge^4 TM, TM). \tag{3.12}$$

On the other hand, recall that the Bianchi forms in the sense of Tachibana are defined by

$$\Omega^{(p)}_{\alpha_1,\dots,\alpha_{2p}} = \Omega^{\alpha_2}_{\alpha_1} \wedge \Omega^{\alpha_3}_{\alpha_2} \wedge \cdots \wedge \Omega^{\alpha_{2p}}_{\alpha_{2p-1}}, \tag{3.13}$$

where $\Omega^{\alpha_{q+1}}_{\alpha_q}$ are 2-forms. Thus, setting

$$S = S^1 \wedge S^2 \wedge \cdots \wedge S^{2m}, \tag{3.14}$$

we find that

$$dS = 4m\alpha \wedge S. \tag{3.15}$$

Therefore, we may say that the $4m$-form S associated with M is ER with $4m\alpha$ as recurrence form.

By (3.4) we may set

$$S^A = u \wedge \omega^A \tag{3.16}$$

and by (3.1) and the structure equations (2.5) we get after some calculations

$$\Theta^A_B = 2(u+\alpha) \wedge \omega^A_B + 2t\omega^B \wedge \omega^A. \tag{3.17}$$

Next, performing the exterior differentiation of Θ^A_B, we derive, taking account of (3.8)

$$d\Theta^A_B = 2(u+\alpha) \wedge \Theta^A_B. \tag{3.18}$$

This shows that all curvature forms Θ^A_B are ER and have the closed 1-form $2(u+\alpha)$ as recurrence form.

We agree to define such an even-dimensional manifold M as an *exterior recurrent curvature* 2*-form manifold*.

Finally, assume that M carries an almost symplectic form Ω. Then, we may express Ω as

$$\Omega = \sum_{a=1}^{m} \omega^a \wedge \omega^{a^*}, \quad a^* = a+m. \tag{3.19}$$

Taking the exterior differentiation of Ω, we find by (3.4) and (3.16) that

$$d\Omega = 2(\alpha + u) \wedge \Omega. \tag{3.20}$$

This shows that the manifold under consideration is endowed with a *conformal* symplectic structure having $\alpha + u$ as covector of Lee.

Moreover, taking the Lie differentiation of Ω with respect to the structure vector field $\mathcal{T}$, we infer

$$\mathcal{L}_{\mathcal{T}}\Omega = ut\Omega + 2(u+\alpha) \wedge \sum_{a=1}^{m} (T^a \omega^{a^*} - T^{a^*} \omega^a). \tag{3.21}$$

Using (3.8) and (3.6), the exterior differentiation of (3.21) gives

$$d\mathcal{L}_{\mathcal{T}}\Omega = 8tu \wedge \Omega. \tag{3.22}$$

Hence, by [4], the above equation says that $\mathcal{T}$ defines a *relative conformal transformation* of the conformal symplectic form Ω.

Summing up, we state the following theorem.

Theorem 3.1. *Let (M,g) be a $2m$-dimensional Riemannian manifold structured by an affine connection defined by the torsion 2-forms S^A, $A = 1,\dots,2m$. Let $\mathcal{T}(T^A)$ be a structure vector field, which defines a $\mathcal{T}$-parallel connection and assume that S^A are exterior recurrent, having $\mathcal{T}^\flat$ as recurrence form ($\mathcal{T}^\flat = \alpha$ is a closed Pfaffian).*

Then the following properties hold:

(i) *$\mathcal{T}$ is both a torse forming and an exterior concurrent vector field;*

(ii) *the structure curvature 2-forms Θ_B^A are exterior recurrent with the closed Pfaffian $2(u+\alpha)$ as recurrence form;*

(iii) *the manifold M is endowed with a conformal symplectic structure Ω having $u+\alpha$ as covector of Lee;*

(iv) *the vector field $\mathcal{T}$ defines a relative conformal transformation of Ω, that is, $d\mathcal{L}_{\mathcal{T}}\Omega = 8tu \wedge \Omega$, where $2t = \|\mathcal{T}\|^2$.*

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