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AN APPLICATION TO KATO'S SQUARE ROOT PROBLEM

TOKA DIAGANA

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We find all complex potentials Q such that the general Schrödinger operator on $\mathbb{R}^n$, given by $L = -\Delta + Q$, where Δ is the Laplace differential operator, verifies the well-known Kato's square problem. As an application, we will consider the case where $Q \in L^1_{\text{loc}}(\Omega)$.

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1. Introduction. Let $\Omega \subseteq \mathbb{R}^n$, an open set and let's stand in the Hilbert space $H = L^2(\Omega, C)$ $(= L^2(\Omega))$. Consider Q, a measurable complex function and let Φ and Ψ be the sesquilinear forms given by,

$$\begin{aligned} \Phi(u,v) &= \int_\Omega \nabla u \overline{\nabla v}\, dx \quad \forall u,v \in D(\phi) = H^1_0(\Omega), \\ \Psi(u,v) &= \int_\Omega Q u \bar{v}\, dx \quad \forall u,v \in D(\Psi), \end{aligned} \tag{1.1}$$

where $D(\Psi) = \{u \in L^2(\Omega) : Q|u|^2 \in L^1(\Omega)\}$. Assume that the potential Q verifies that there exists $\beta > 0$ and there exists $\theta \in (0, \pi/2)$, such that

$$|\arg(Q-\beta)| \le \frac{\pi}{2} - \theta. \tag{1.2}$$

The sesquilinear forms Φ and Ψ are both closed, densely defined, and sectorial. According to Kato's first representation theorem (see [2]), we can associate to Φ and Ψ, m-sectorial linear operators defined, respectively, by

$$\begin{aligned} Au &= -\Delta u \quad \text{with} \quad D(A) = \{u \in H^1_0(\Omega) : \Delta u \in L^2(\Omega)\}, \\ Bu &= Qu \quad \text{with} \quad D(B) = \{u \in L^2(\Omega) : Qu \in L^2(\Omega)\}. \end{aligned} \tag{1.3}$$

By Schrödinger operator, we mean a partial differential operator on $\mathbb{R}^n$ of the form

$$L = A + B; \quad A = -\Delta; \quad B = Q = Q(x), \tag{1.4}$$

where Δ is the n-dimensional Laplace operator $\Delta = \sum_{i=1}^n \partial^2/\partial x_i^2$. The name comes from the form of Schrödinger's equation which, in units with $h = m = 1$ reads

$$i\frac{\partial u}{\partial t} = Lu. \tag{1.5}$$

Our aim here is to find all potentials Q such that

$$D(L^{1/2}) = D(A^{1/2}) \cap D(B^{1/2}) = D(L^{*1/2}). \tag{1.6}$$

For that we use the author results [1] related to the sum of linear operators connected to Kato's square root problem. The case where $\Omega = \mathbb{R}^n$ will be studied later as a consequence of the general case.

2. Schrödinger operators and Kato's condition

Definition 2.1. A linear operator C is said to verify Kato's square root problem (or Kato's condition) if

$$D(C^{1/2}) = D(\Upsilon) = D(C^{*1/2}), \tag{2.1}$$

where Υ is the sesquilinear form associated to C.

Hypothesis on Q. Suppose that Q is chosen such as,

$$\overline{D(\Phi)\cap D(\Psi)} = L^2(\Omega). \tag{2.2}$$

Proposition 2.2. *Let A and B be the linear operators given by (1.3). Assume that the potential Q verifies (2.2). Then there exists a unique operator sum $A\oplus B$, which is m-sectorial, verifying Kato's condition and*

(i) $A\oplus B = \overline{A+B}$ *if* $\overline{A+B}$ *is a maximal operator,*

(ii) $|\operatorname{Im}\langle (A\oplus B)u,u\rangle| \le \operatorname{Re}\langle (A\oplus B)u,u\rangle$, *for all* $u\in D(A\oplus B)$.

Proof. Assume that Q verifies hypothesis (2.2). So, the sesquilinear form given by, $\Upsilon = \Phi+\Psi$, is a closed, sectorial, and densely defined. By Kato's first representation theorem (see [2]), there exists a unique m-sectorial sum operator, $A\oplus B$, associated to Υ, verifying

$$\Upsilon(u,v) = \langle (A\oplus B)u,v\rangle \quad \forall u\in D(A\oplus B),\ v\in D(\Phi)\cap D(\Psi) \tag{2.3}$$

since A and B both verify Kato's condition, according to author's theorem (see [1, Theorem 2, page 462]), the operator $A\oplus B$ verifies the same condition, that is,

$$D\big((A\oplus B)^{1/2}\big) = D(\Phi)\cap D(\Psi) = D\big((A\oplus B)^{*1/2}\big) \tag{2.4}$$

and (ii) is satisfied, where β is given by (1.2). The operator $A\oplus B$ is defined as

$$\begin{aligned} &(A\oplus B)u = -\Delta u + Qu, \quad \forall u\in D(A\oplus B),\\ &D(A\oplus B) = \{u\in H_0^1(\Omega) : Q|u|^2\in L^1(\Omega),\ -\Delta u+Qu\in L^2(\Omega)\}. \end{aligned} \tag{2.5}$$

Using also author's theorem (see [1, Theorem 2, page 462]), it follows that (i) is satisfied. □

3. Some applications. Consider the same operators, that is, $A = -\Delta$ and $B = Q$ in $L^2(\Omega)$. Assume that $Q\in L^1_{\text{loc}}(\Omega)$, in this case (2.2) is satisfied. According to Brézis and Kato, the operator $\overline{A+B}$ is maximal in $L^2(\Omega)$ (then $A\oplus B = \overline{A+B}$ and Kato's condition is satisfied) and is given by (2.5).

Case $\Omega = \mathbb{R}^n$**.** We always assume $Q \in L^1_{loc}(\mathbb{R}^n)$, it follows that

$$\begin{aligned} Au &= -\Delta u \quad \text{with} \quad D(A) = H^2(\mathbb{R}^n); \qquad D(A^{1/2}) = H^1(\mathbb{R}^n), \\ Bu &= Qu \quad \text{with} \quad D(B^{1/2}) = \{u \in L^2(\mathbb{R}^n) : Q|u|^2 \in L^1(\mathbb{R}^n)\}, \end{aligned} \tag{3.1}$$

and $D(A^{1/2}) \cap D(B^{1/2})$ is dense in $L^2(\mathbb{R}^n)$ because,

$$C_0^\infty(\mathbb{R}^n) \subseteq D(\Phi) \cap D(\Psi). \tag{3.2}$$

In conclusion, Kato's condition is satisfied by $\overline{A+B}$, that is,

$$D\left(\sqrt{\overline{A+B}}\right) = D(\sqrt{A}) \cap D(\sqrt{B}) = D\left(\sqrt{\overline{A+B}^*}\right). \tag{3.3}$$

For example when $n = 1$, then

$$D\left(\sqrt{\overline{A+B}}\right) = H^1(\mathbb{R}) = D\left(\sqrt{\overline{A+B}^*}\right). \tag{3.4}$$

Remark 3.1. Condition (2.2) could be weakened as

$$\overline{D(A) \cap D(B)} = L^2(\Omega). \tag{3.5}$$

But in general the algebraic sum of two operators is not always defined (because this concept is not well adapted to problems arising in mathematical analysis).

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References

[1] T. Diagana, *Sommes d'opérateurs et conjecture de Kato-McIntosh* [*Sums of linear operators and the Kato-McIntosh conjecture*], C. R. Acad. Sci. Paris Sér. I Math. **330** (2000), no. 6, 461–464 (French).

[2] T. Kato, *Perturbation Theory for Linear Operators*, Die Grundlehren der mathematischen Wissenschaften, vol. 132, Springer-Verlag, New York, 1966.

Toka Diagana: Department of Mathematics, Howard University, 2441, Sixth Street, NW Washington, DC 20059, USA

E-mail address: tdiagana@howard.edu