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ON THE CONTINUITY OF PRINCIPAL EIGENVALUES FOR BOUNDARY VALUE PROBLEMS WITH INDEFINITE WEIGHT FUNCTION WITH RESPECT TO RADIUS OF BALLS IN $\mathbb{R}^N$

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We investigate the continuity of principal eigenvalues (i.e., eigenvalues corresponding to positive eigenfunctions) for the boundary value problem $-\Delta u(x)=\lambda g(x)u(x)$, $x\in B_R(0)$; $u(x)=0$, $|x|=R$, where $B_R(0)$ is a ball in $\mathbb{R}^N$, and g is a smooth function, and we show that $\lambda_1^+(R)$ and $\lambda_1^-(R)$ are continuous functions of R.

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1. Introduction. We study the function $R\to\lambda_1^+(R)$ where $\lambda_1^+(R)$ being the unique positive principal eigenvalue (i.e., eigenvalue corresponding to positive eigenfunction) for the boundary value problem

$$-\Delta u(x)=\lambda g(x)u(x),\quad x\in B_R(0);\qquad u(x)=0,\quad |x|=R, \tag{1.1}$$

where Δ is the standard Laplace operator, $B_R(0)$ is a ball in $\mathbb{R}^N$, and $g:B_R(0)\to R$ is a smooth function with changes in sign on $B_R(0)$.

Recently, there has been interest in such problems since Fleming [4] studied the following equation which arises in population genetics:

$$u_t(x,t)=\Delta u+\lambda g(x)f(u),\quad x\in D, \tag{1.2}$$

where D is a bounded domain with smooth boundary, g changes sign on D, and f is some function of class C^1 such that $f(0)=0=f(1)$.

Fleming's results suggested that nontrivial steady-state solutions were bifurcating the trivial solutions $u\equiv 0$ and $u\equiv 1$. In order to investigate these bifurcation phenomena, it was necessary to understand the eigenvalues and eigenfunctions of the corresponding linearized problem

$$-\Delta u(x)=\lambda g(x)u(x),\quad x\in D. \tag{1.3}$$

The ordinary differential equation versions of (1.3) were studied by Picone [7] and Bocher [2]. Motivated by Fleming's paper, Brown and Lin [3], and Hess and Kato [5] studied the eigenvalues and eigenfunctions of (1.3) in the partial differential equation case. Since, in population genetics, the unknown function u represents the frequency of a population, only the solutions $u\geq 0$ are of interest.

In order that nontrivial solutions bifurcating the zero solution are positive, it is necessary that the eigenfunction of the corresponding eigenvalue is positive. Such eigenvalues and eigenfunctions are called principal eigenvalues and eigenfunctions.

The existence of principal eigenvalues of (1.1) has been studied previously in [3, 5, 6]. It is well known (see [5]) that there exists a double sequence of eigenvalues for (1.1)

$$\cdots < \lambda_2^- < \lambda_1^- < 0 < \lambda_1^+ < \lambda_2^+ < \cdots, \tag{1.4}$$

$\lambda_1^+(\lambda_1^-)$ being the unique positive (negative) principal eigenvalue, that is, (1.1) has solutions $u(v)$ which are positive in $B_R(0)$, and we call $u(v)$ principal eigenfunction corresponding to principal eigenvalue $\lambda_1^+(\lambda_1^-)$.

The variational characterizations of $\lambda_1^+(R)$ and $\lambda_1^-(R)$ are proved in [1].

Theorem 1.1. *The variational characterizations of $\lambda_1^+(R)$ and $\lambda_1^-(R)$ are given by*

$$\begin{aligned} \lambda_1^+(R) &= \inf\left\{\frac{\int_{B_R(0)} |\nabla u|^2 dx}{\int_{B_R(0)} g(x)u^2 dx} : u \in H_0^1(B_R(0)),\ \int_{B_R(0)} g(x)u^2 dx > 0\right\}, \\ \lambda_1^-(R) &= \sup\left\{\frac{\int_{B_R(0)} |\nabla u|^2 dx}{\int_{B_R(0)} g(x)u^2 dx} : u \in H_0^1(B_R(0)),\ \int_{B_R(0)} g(x)u^2 dx < 0\right\}. \end{aligned} \tag{1.5}$$

The following theorem is proved in [1].

Theorem 1.2. *The characterization of $\lambda_1^+(R)$ is given by*

$$\lambda_1^+(R) = \inf\left\{\frac{\int_{B_R(0)} |\nabla u|^2 dx}{\int_{B_R(0)} g(x)u^2 dx} : u \in C_0^\infty(B_R(0)),\ \int_{B_R(0)} g(x)u^2 dx > 0\right\}, \tag{1.6}$$

and similarly for $\lambda_1^-(R)$.

2. On the continuity of $\lambda_1^+(R)$ and $\lambda_1^-(R)$ with respect to R. First, we prove that $\lambda_1^+(R)$ $(\lambda_1^-(R))$ is a strictly decreasing (increasing) function of R.

Theorem 2.1. *The function $R \to \lambda_1^+(R)$ is a strictly decreasing function.*

Proof. It is proved in [1] that $R \to \lambda_1^+(R)$ is a decreasing function of R, so it is sufficient to show its strictness. We prove it by a contradiction argument. On the contrary, suppose that there exists R and R' such that $R < R'$ but $\lambda_1^+(R) = \lambda_1^+(R')$. Then there exist two positive functions u and v on $B_R(0)$ and $B_{R'}(0)$, respectively, such that

$$-\Delta u(x) = \lambda_1^+(R)g(x)u(x), \quad x \in B_R(0); \qquad u(x) = 0, \quad |x| = R, \tag{2.1}$$

$$-\Delta v(x) = \lambda_1^+(R')g(x)v(x), \quad x \in B_{R'}(0); \qquad v(x) = 0, \quad |x| = R'. \tag{2.2}$$

From (2.2) we have

$$-\Delta v(x) = \lambda_1^+(R')g(x)v(x), \quad x \in B_R(0); \qquad v(x) > 0, \quad |x| = R. \tag{2.3}$$

Multiplying (2.1) by v and integrating over $B_R(0)$, we obtain

$$\int_{B_R(0)} \nabla u(x)\nabla v(x)dx - \int_{|x|=R} v(x)\frac{\partial u}{\partial n}(x)ds = \lambda_1^+(R)\int_{B_R(0)} g(x)u(x)v(x)dx, \tag{2.4}$$

also multiplying (2.3) by u and integrating over $B_R(0)$ we obtain

$$\int_{B_R(0)} \nabla u(x)\nabla v(x)dx-\int_{|x|=R} u(x)\frac{\partial v}{\partial n}(x)ds=\lambda_1^+(R')\int_{B_R(0)} g(x)u(x)v(x)dx. \quad (2.5)$$

Now by subtracting (2.5) from (2.4) we obtain

$$-\int_{|x|=R} v(x)\frac{\partial u}{\partial n}(x)ds = \left[\lambda_1^+(R)-\lambda_1^+(R')\right]\int_{B_R(0)} g(x)u(x)v(x)dx. \quad (2.6)$$

By (2.1) and (2.3) we have

$$\int_{|x|=R} v(x)\frac{\partial u}{\partial n}(x)ds<0, \quad (2.7)$$

and so by (2.6) we obtain $\lambda_1^+(R)-\lambda_1^+(R')\neq 0$, and this is a contradiction. □

Also by a similar argument we can obtain the following results.

Theorem 2.2. *The function $\lambda_1^-(R)$ is a strictly increasing function of R.*

Theorem 2.3. *The function $R\to\lambda_1^+(R)$ is a continuous function of R.*

Proof. Let $\epsilon>0$ be given. Let $R_1>R$ and sufficiently close to R, it is enough to show that

$$\lambda_1^+(R)<\lambda_1^+(R_1)+\epsilon. \quad (2.8)$$

Let $\varphi_1\in H_0^1(B_{R_1}(0))$ be such that

$$-\Delta\varphi_1(x)=\lambda_1^+(R_1)g(x)\varphi_1(x),\quad x\in B_{R_1}(0);\qquad \varphi_1(x)=0,\quad |x|=R_1. \quad (2.9)$$

We define $y=(R_1/R)x$ and $\hat{\varphi}(x)=\varphi_1(y)$ for $x\in B_R(0)$. We have $\hat{\varphi}\in H_0^1(B_R(0))$ and we have

$$\begin{aligned}
&\left|\int_{B_{R_1}(0)} g(x)\varphi_1^2(x)dx-\int_{B_R(0)} g(x)\hat{\varphi}^2(x)dx\right| \\
&\quad=\left|\int_{B_R(0)} g(x)\varphi_1^2(x)dx-\int_{B_R(0)} g(x)\hat{\varphi}^2(x)dx+\int_{B_{R_1}(0)-B_R(0)} g(x)\varphi_1^2(x)dx\right| \\
&\quad\le\left|\int_{B_R(0)} g(x)\varphi_1^2(x)dx-\int_{B_R(0)} g(x)\hat{\varphi}^2(x)dx\right|+\left|\int_{B_{R_1}(0)-B_R(0)} g(x)\varphi_1^2(x)dx\right| \\
&\quad\le\int_{B_R(0)}|g(x)|\,|\varphi_1^2(x)-\hat{\varphi}^2(x)|\,dx+\int_{B_{R_1}(0)-B_R(0)}|g(x)\varphi_1^2(x)|\,dx \\
&\quad\le\left(\int_{B_R(0)}\left|\varphi_1^2(x)-\varphi_1^2\left(\frac{R_1}{R}x\right)\right|dx\right)\sup_{x\in B_R(0)}|g(x)| \\
&\qquad+|B_{R_1}(0)-B_R(0)|\sup_{x\in B_{R_1}(0)-B_R(0)}|g(x)\varphi_1^2(x)|.
\end{aligned} \quad (2.10)$$

Since $\int_{B_R(0)} |\varphi_1^2(x) - \varphi_1^2((R_1/R)x)|dx \to 0$ and $|B_{R_1}(0) - B_R(0)| \to 0$ as $R_1 \to R$, we have

$$\int_{B_R(0)} g(x)\hat{\varphi}^2(x)dx > 0. \tag{2.11}$$

So

$$\begin{aligned}\lambda_1^+(R) &\le \frac{\int_{B_R(0)} |\nabla\hat{\varphi}(x)|^2 dx}{\int_{B_R(0)} g(x)\hat{\varphi}^2(x)dx} \\ &= \left(\frac{R_1}{R}\right)^2 \frac{\int_{B_{R_1}(0)} |\nabla\varphi_1(y)|^2 dy}{\int_{B_{R_1}(0)} g(y)\varphi_1^2(y)dy} \\ &= \left(\frac{R_1}{R}\right)^2 \lambda_1^+(R_1) \\ &< \lambda_1^+(R_1) + \epsilon.\end{aligned} \tag{2.12}$$

The last inequality holds if we choose R_1 such that $R_1 > R$ and sufficiently close to R. If R_1 is chosen such that for every $R' \in (R, R_1)$, we have

$$\lambda_1^+(R) - \lambda_1^+(R') < \lambda_1^+(R) - \lambda_1^+(R_1) < \epsilon. \tag{2.13}$$

Hence $\lambda_1^+(R)$ is a continuous function of R. □

Also by a quite similar argument we can prove the following theorem.

Theorem 2.4. *The function $R \to \lambda_1^-(R)$ is a continuous function of R.*

Theorem 2.5. *Let $\lambda_1^-(R) < \lambda < \lambda_1^+(R)$, then there exists $R' > R$ such that*

$$\lambda_1^-(R') < \lambda < \lambda_1^+(R'). \tag{2.14}$$

Proof. Let $\epsilon = \lambda_1^+(R) - \lambda$, so $\epsilon > 0$. By using the continuity of the function $R \to \lambda_1^+(R)$, there exists $R_1 > R$ such that $\lambda_1^+(R) - \lambda_1^+(R_1) < \epsilon$. Then we have

$$\lambda < \lambda_1^+(R_1). \tag{2.15}$$

Similarly with the continuity of the function $R \to \lambda_1^-(R)$, there exists $R_2 > R$ such that

$$\lambda > \lambda_1^-(R_2). \tag{2.16}$$

Now let $R' = \min\{R_1, R_2\}$, we have

$$\lambda_1^-(R') \le \lambda_1^-(R_2) < \lambda < \lambda_1^+(R_1) \le \lambda_1^+(R'), \tag{2.17}$$

and so the proof is complete. □

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