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SOME SEQUENCE SPACES AND STATISTICAL CONVERGENCE

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We introduce the strongly (V,λ)-convergent sequences and give the relation between strongly (V,λ)-convergence and strongly (V,λ)-convergence with respect to a modulus.

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1. Introduction. Let $\lambda = (\lambda_n)$ be a nondecreasing sequence of positive numbers tending to ∞, and $\lambda_{n+1} \le \lambda_n + 1$, $\lambda_1 = 1$.

The generalized de la Vallée-Poussin mean is defined by

$$t_n = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k, \tag{1.1}$$

where $I_n = [n - \lambda_n + 1, n]$. A sequence $x = (x_k)$ is said to be (V,λ)-summable to a number L (see [5]) if $t_n(x) \to L$ as $n \to \infty$. If $\lambda_n = n$, then (V,λ)-summability is reduced to $(C,1)$-summability. We write

$$[V,\lambda] = \left\{ x = (x_k) : \text{ for some } L,\ \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k - L| = 0 \right\} \tag{1.2}$$

for sets of sequences $x = (x_k)$ which are strongly (V,λ)-summable to L, that is, $x_k \to L[V,\lambda]$.

We recall that a modulus f is a function from $[0,\infty)$ to $[0,\infty)$ such that

(i) $f(x) = 0$ if and only if $x = 0$;
(ii) $f(x+y) \le f(x) + f(y)$ for all $x, y \ge 0$;
(iii) f is increasing;
(iv) f is continuous from the right at 0.

It follows that f must be continuous on $[0,\infty)$. A modulus may be bounded or unbounded. Maddox [6] and Ruckle [9] used the modulus f to construct sequence spaces. In this paper, we introduce the strongly (V,λ)-convergent sequences and give the relation between strongly (V,λ)-convergence and strongly (V,λ)-convergence with respect to a modulus.

2. Some sequence spaces

Definition 2.1. Let f be a modulus. We define the spaces,

$$\begin{aligned} [V,\lambda,f] &= \left\{ x = (x_k) : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} f(|x_k - L|) = 0, \text{ for some } L \right\}, \\ [V,\lambda,f]_0 &= \left\{ x = (x_k) : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} f(|x_k|) = 0 \right\}. \end{aligned} \tag{2.1}$$

When $\lambda_n = n$ then the sequence spaces defined above become $w_0(f)$ and $w(f)$, respectively, where $w_0(f)$ and $w(f)$ are defined by Maddox [6].

Note that if we put $f(x) = x$, then we have $[V,\lambda,f] = [V,\lambda]$ and $[V,\lambda,f]_0 = [V,\lambda]_0$, where

$$[V,\lambda]_0 = \left\{ x = (x_k) : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k| = 0 \right\}. \tag{2.2}$$

We have the following result.

Theorem 2.2. *The spaces $[V,\lambda,f]$ and $[V,\lambda,f]_0$ are linear spaces.*

Proof. We consider only $[V,\lambda,f]$. Suppose that $x_i \to L$ and $y_j \to L'$ in $[V,\lambda,f]$ and that α, β are in $\mathbb{C}$. Then there exists integers T_α and M_β such that $|\alpha| \le T_\alpha$ and $|\beta| \le M_\beta$. We therefore have

$$\begin{aligned} &\frac{1}{\lambda_n} \sum_{k \in I_n} f\left(\left|\alpha x_k + \beta x_k - (\alpha L + \beta L')\right|\right) \\ &\quad \le T_\alpha \frac{1}{\lambda_n} \sum_{k \in I_n} f\left(\left|x_k - L\right|\right) + M_\beta \frac{1}{\lambda_n} \sum_{k \in I_n} f\left(\left|x_k - L'\right|\right). \end{aligned} \tag{2.3}$$

This implies that $\alpha x + \beta y \to \alpha L + \beta L'$ in $[V,\lambda,f]$. This completes the proof. □

Proposition 2.3 (see [7]). *Let f be any modulus. Then $\lim_{t\to\infty} f(t)/t = \beta$ exists.*

Proposition 2.4. *Let f be a modulus and let $0 < \delta < 1$. Then for each $x \ge \delta$ we have $f(x) \le 2f(1)\delta^{-1}x$.*

This can be proved by using the techniques similar to those used in Maddox [6] and hence we omit the proof.

Theorem 2.5. *Let f be any modulus. If $\lim_{t\to\infty} f(t)/t = \beta > 0$, then $[V,\lambda,f] = [V,\lambda]$.*

Proof. If $x \in [V,\lambda]$, then

$$s_n = \frac{1}{\lambda_n} \sum_{k \in I_n} \left|x_k - L\right| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \text{ for some } L. \tag{2.4}$$

Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \varepsilon$ for every t with $0 \le t \le \delta$. We can write

$$\begin{aligned} \frac{1}{\lambda_n} \sum_{k \in I_n} f\left(\left|x_k - L\right|\right) &= \frac{1}{\lambda_n} \sum_{k \in I_n, |x_k - L| \le \delta} f\left(\left|x_k - L\right|\right) + \frac{1}{\lambda_n} \sum_{k \in I_n, |x_k - L| > \delta} f\left(\left|x_k - L\right|\right) \\ &\le \frac{1}{\lambda_n}\left(\lambda_n \cdot \varepsilon\right) + 2f(1)\delta^{-1} s_n, \end{aligned} \tag{2.5}$$

by Proposition 2.4, as $n \to \infty$. Therefore $x \in [V,\lambda,f]$. It is trivial that $[V,\lambda,f] \subset [V,\lambda]$ and this completes the proof. □

3. λ-statistical convergence. In [3], Fast introduced the idea of statistical convergence, which is closely related to the concept of natural density or asymptotic density of subsets of the positive integers $\mathbb{N}$. In recent years, statistical convergence has been studied by several authors [1, 2, 4, 8, 10].

A sequence $x = (x_k)$ is said to be statistically convergent to the number L if for every $\varepsilon > 0$,

$$\lim_n \frac{1}{n} \left| \{k \le n : |x_k - L| \ge \varepsilon\} \right| = 0, \tag{3.1}$$

where the vertical bars indicate the number of elements in the enclosed set. In this case we write $s - \lim x = L$ or $x_k \to L(s)$ and s denotes the set of all statistically convergent sequences.

In this section, we introduce and study the concept of λ-statistical convergence and find its relation with $[V,\lambda,f]$ and s_λ.

Definition 3.1. A sequence $x = (x_k)$ is said to be λ-statistically convergent or s_λ-convergent to L if for every $\varepsilon > 0$,

$$\lim_n \frac{1}{\lambda_n} \left| \{k \in I_n : |x_k - L| \ge \varepsilon\} \right| = 0. \tag{3.2}$$

In this case, we write $s_\lambda - \lim x = L$ or $x_k \to L(s_\lambda)$ and $s_\lambda = \{x : \text{ for some } L,\ s_\lambda - \lim x = L\}$. Note that if $\lambda_n = n$, then s_λ is same as s.

The following definition was introduced by Connor [2] as an extension of the original definition of statistical convergence which appeared in [3].

Definition 3.2. Let A be a nonnegative regular summability method and let x be a sequence. Then x is said to be A-statistically convergent to L if $\chi_{S(x-Le:\varepsilon)}$ is contained in $w_0(A)$ for every $\varepsilon > 0$, where

$$w_0(A) = \left\{x : \lim_n \sum a_{n,k} |x_k| = 0\right\}. \tag{3.3}$$

In the above definition, if we define the matrix by

$$a_{n,k} = \begin{cases} \dfrac{1}{\lambda_n}, & \text{if } n \in I_n, \\ 0, & \text{if } n \notin I_n \end{cases} \tag{3.4}$$

we get λ-statistical convergence as a special case of A-statistical convergence.

Let ∇ denote the set of all nondecreasing sequences $\lambda = (\lambda_n)$ of positive numbers tending to ∞ such that $\lambda_{n+1} \le \lambda_n + 1$ and $\lambda_1 = 1$.

We have the following result.

Theorem 3.3. *Let $\lambda \in \nabla$ and f be any modulus. Then $[V,\lambda,f] \subset (s_\lambda)$.*

Proof. Suppose that $\varepsilon > 0$ and $x \in [V,\lambda,f]$. Since,

$$\begin{aligned}\frac{1}{\lambda_n}\sum_{k\in I_n} f(|x_k-L|) &\geq \frac{1}{\lambda_n}\sum_{k\in I_n,|x_k-L|\geq\varepsilon} f(|x_k-L|)\\ &\geq \frac{1}{\lambda_n} f(\varepsilon)\cdot |\{k\in I_n : |x_k-L|\geq\varepsilon\}|\end{aligned} \tag{3.5}$$

from which it follows that $x \in (s_\lambda)$. This completes the proof. □

Theorem 3.4. $(s_\lambda) = [V,\lambda,f]$ *if and only if* f *is bounded.*

Proof. Suppose that f is bounded and that $x \in (s_\lambda)$. Since f is bounded, there is a constant M such that $f(x) \leq M$ for all $x \geq 0$. Given $\varepsilon > 0$, we have

$$\begin{aligned}\frac{1}{\lambda_n}\sum_{k\in I_n} f(|x_k-L|) &\leq \frac{1}{\lambda_n}\sum_{k\in I_n,|x_k-L|\geq\varepsilon} f(|x_k-L|) + \frac{1}{\lambda_n}\sum_{k\in I_n,|x_k-L|<\varepsilon} f(|x_k-L|)\\ &\leq \frac{M}{\lambda_n}|\{k\in I_n : |x_k-L|\geq\varepsilon\}| + f(\varepsilon).\end{aligned} \tag{3.6}$$

Taking the limit as $\varepsilon \to 0$, the result follows. Conversely, suppose that f is unbounded so that there is a positive sequence $0 < t_1 < t_2 < \cdots < t_i < \cdots$ such that $f(t_i) \geq \lambda_i$. Define the sequence $x = (x_i)$ by putting $x_{k_i} = t_i$ for $i = 1,2,\ldots$ and $x_i = 0$ otherwise. Then we have $x \in (s_\lambda)$, but $x \notin [V,\lambda,f]$. □

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