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NUMERICAL AND GRAPHICAL DESCRIPTION OF ALL AXIS CROSSING REGIONS FOR THE MODULI 4 AND 8 WHICH OCCUR BEFORE 10^{12}

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ABSTRACT. Let $\pi_{b,c}(x)$ denote the number of primes $\leq x$ and $\equiv c \pmod b$, and for positive integers x let $\Delta_b(x,c,1) = \pi_{b,c}(x) - \pi_{b,1}(x)$. Negative values of $\Delta_4(x,3,1)$ less than 10^{12} occur in six widely spaced regions. The first three regions, investigated by Leech [6], Shanks [9], and Lehmer [6], contain only a few thousand negative values of $\Delta_4(x,3,1)$. However, the authors [1] have recently discovered 3 new regions, the sixth occurring before 20 billion and containing more than half a billion negative values of $\Delta_4(x,3,1)$. In this paper numerical and graphical details of all six regions are given. Moreover, new results for the modulus 8 are presented. Previously, no negative values have been found for $\Delta_8(x,c,1)$, c = 3,5, or 7 and our search to 10^{12} reveals no such values for $\Delta_8(x,3,1)$ or $\Delta_8(x,7,1)$. For $\Delta_8(x,5,1)$ we have discovered the first two regions of negative values. The first of these regions, beginning at x = 588067889, contains

422,500 negative values of $\Delta_8(x,5,1)$; the second occurs in the vicinity of 35 billion and contains more than a billion negative values of $\Delta_8(x,5,1)$.

KEY WORDS AND PHRASES. *Prime Numbers, Quadratic Non-residue, and Axis Crossing Regions.*

AMS (MOS) SUBJECT CLASSIFICATION (1970) CODES. *Primary 1003, 1005, 1008.*

1. INTRODUCTION.

Let $\pi_{b,c}(x)$ denote the number of primes $\le x$ in the arithmetic progressions $bn + c$, $1 \le c < b$, $(b,c) = 1$. For moduli b having exactly one real nonprincipal character (i.e., b = 3,4,6,8,12, and 24) and for quadratic non-residues c, let

$$\Delta_b(x,c,1) = \pi_{b,c}(x) - \pi_{b,1}(x) \quad , \tag{1.1}$$

Due to the famous work of J. E. Littlewood [8] it is known that $\Delta_4(x,3,1)$ and $\Delta_6(x,5,1)$ assume negative values infinitely often. Knowledge regarding sign changes of $\Delta_b(x,c,1)$ has expanded considerably recently - see, for example, Knapowski and Turán [4], [5].

Somewhat surprisingly, in light of the century long interest in the phenomenon initiated by remarks of Chebyshev [3], it was not until 1957 that John Leech discovered that $\Delta_4(x,3,1)$ assumes a negative value for the first time (called a first axis crossing) at x = 26,861. Leech [6] and Shanks [9] found 3404 negative values of $\Delta_4(x,3,1)$ less than $3 \cdot 10^6$.

Based on the numerical work of Leech, of Shanks, and later work of Lehmer [7] it was believed until recently that negative values of $\Delta_b(x,c,1)$ must occur exceedingly rarely. However, the authors [1] recently discovered three new axis crossing regions for b = 4 including a thoroughly remarkable region occuring before 20 billion which contains more than half a billion integers x with $\Delta_4(x,3,1)$ negative. Our opinion that this region represents a typical state of affairs is reinforced by our discovery (see Table 1) of 1,251,299,196 negative values of

$\Delta_8(x,5,1)$ occuring between 35,615,130,497 and 37,335,021,852 whereas only 422,500 negative values occur before 35,615,130,497.

2. NOTATION AND PRELIMINARIES.

Integers x for which $\Delta_b(x,c,1)$ is negative, zero, or positive will be said to lie respectively below, on, or above the axis.

Analogous to our definition in [1] we define the ℓ-th axis crossing region for each $\ell > 1$ and fixed progression bn + c to be the ℓ-th set of consecutive positive integers $x_0(\ell)$, $x_1(\ell),\ldots,x_f(\ell)$ with the properties that (1),

$$\Delta_b(x_0(\ell),c,1) \;=\; \Delta_b(x_f(\ell),c,1) = -1 \quad , \tag{2.1}$$

and that (2) $\Delta_b(x,c,1) \geq 0$ for each integer x with $x_0(\ell) > x > x_f(\ell-1)$ and $x_0(\ell) > 2x_f(\ell-1)$. We call $x_0 = x_0(\ell)$ a first regional axis crossing and $x_f = x_f(\ell)$ a last regional axis crossing.

One of the interesting features in the interior of axis crossing regions is the existence of large negative blocks (consecutive integers with $\Delta_b(x,c,1) < 0$). In some cases a single negative block may include the majority of all integers below the axis in an axis crossing region. A non-negative block is a set of consecutive integers x with $\Delta_b(x,c,1) \geq 0$ lying inside an axis crossing region. Clearly the number of negative blocks in a region exceeds the number of non-negative blocks by exactly one.

3. AXIS CROSSING REGIONS FOR b = 4 AND b = 8 TO 10^{12}.

Tables 1 and 2 include numerical details of all regions for the moduli 4 and 8 discovered over the range to 10^{12}. The fluctuations of $\Delta_b(x,c,1)$ for $x > 10^9$ are sufficiently restrained that prime counts at intervals of 10 million, and intra-interval checking when $\Delta_b(x,c,1)$ is small, render it extremely improbable that our program overlooked any axis crossing regions beyond $x = 10^9$. For $x < 10^9$, a check was made at every prime.

The word "value" in Table 1 refers always to values of $\Delta_4(x,3,1)$ for regions 1 to 6 and to $\Delta_8(x,5,1)$ for regions 7 and 8 in table 2. All classifications are in reference to the region from the first -1 value, x_0 (the first regional axis crossing), through the last -1 value, x_f (the last regional axis crossing), except the first and last zero values which are less than x_0 and greater than x_f respectively, and the classifications "total integers on the axis" and "total integers not below the axis" which are in reference to integers between the first and last zero values. It is curious that we have found no negative values of $\Delta_8(x,3,1)$ and $\Delta_8(x,7,1)$ given the relatively small first negative values of $\Delta_4(x,3,1)$ and $\Delta_8(x,5,1)$.

Figures 1 through 8 depict regions 1 through 8 respectively (see Tables 1 and 2). Figures 9 and 10 are detailed plots of regions 4 and 7, since these regions are relatively tiny. Except for figures 9 and 10, all plots give values for 10% of the integers in the vicinity of the particular region. The lower horizontal line is the zero axis and the upper horizontal line approximates $\pi(x^{\frac{1}{2}})/4$. The equally spaced vertical lines give specific values for x and $\Delta_b(x,c,1)$. When $x > 10^8$, (figures 4 - 10) it is given in billions. With the exception of figures 1 and 2, each plot consists of about 2,000 points.

Table 1

Regions for b=4, c=3

Region Number	1	2	3	4	5	6
First zero value > $x_o/2$	26833	616,769	12,306,061	951,784,469	6,309,280,697	18,465,126,217
x_o - the first axis crossing	26861	616,841	12,306,137	951,784,481	6,309,280,709	18,465,126,293
smallest negative value	-1	-8	-24	-48	-1374	-2719
first occurs here	26861	623,681	12,366,589	951,867,937	6,345,026,833	18,699,356,321
largest positive value	—	12	35	132	361	948
first occurs here	—	625,367	12,340,787	952,090,739	6,394,712,131	19,023,144,611
longest negative block	2	410	15,358	41,346	43,233,786	416,889,978
starts here	26861	628,013	12,361,933	951,846,433	6,330,591,013	18,536,693,261
longest non-negative block	—	5094	14,906	267,450	7,083,982	21,600,450
starts here	—	617,719	12,335,111	951,932,567	6,391,421,479	19,010,515,943
total integers below axis	2	3404	27,218	120,308	72,069,430	518,640,600
total integers not below axis	—	13,554	48,972	318,702	21,800,060	49,757,646
total integers on axis	60	1282	2514	7682	137,422	150,058
number of negative blocks	1	52	75	182	3117	3226
ratio: total integers below axis/x_f	.00007	.0054	.0022	.0001	.011	.027
ratio: total integers on axis/x_f	.002	.002	.0002	.000008	.00002	.000008
x_f - the last axis crossing	26862	633,798	12,382,326	952,223,490	6,403,150,198	19,033,524,538
last zero value < $2x_f$	26926	633,882	12,424,002	952,223,506	6,403,150,362	19,033,524,562

Table 2

Regions for b = 8, c = 5

Region Number	7	8
First zero value $x_0/2$	588,065,761	35,615,130,457
x_0 - the first axis crossing	588,067,889	35,615,130,497
smallest negative value	-57	-1945
first occurs here	592,258,217	36,983,065,913
largest positive value	215	1204
first occurs here	593,136,989	37,203,252,469
longest negative block	216,412	505,210,620
starts here	592,239,337	35,826,280,657
longest non-negative block	1,486,740	95,571,860
starts here	589,516,349	35,701,644,389
total integers below axis	422,500	1,251,299,196
total integers not below axis	5,381,200	468,592,160
total integers on axis	22192	1,087,628
number of negative blocks	249	11161
ratio: total integers below axis/x_f	.00071	.034
ratio: total integers on axis/x_f	.00004	.00003
x_f - the last axis crossing	593,871,588	37,335,021,852
last zero value $2x_f$	593,890,740	37,335,022,092

Fig. 1

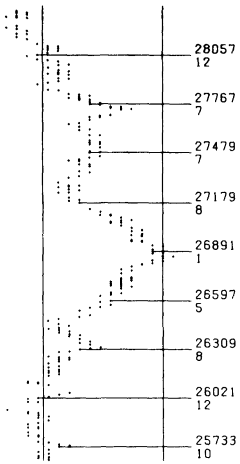

Fig. 2

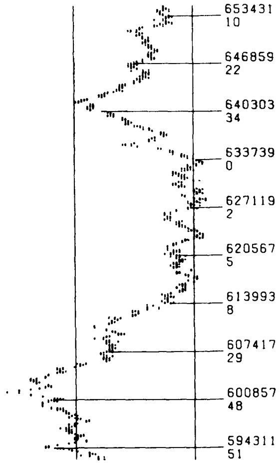

Fig. 3

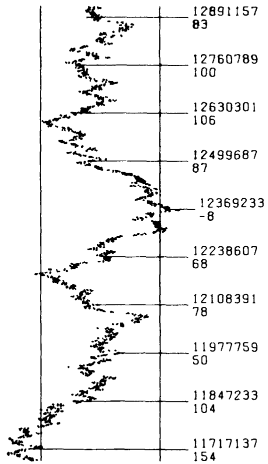

Fig. 4

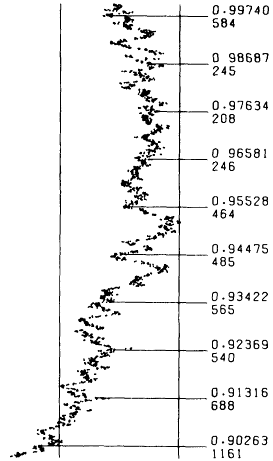

Fig. 5

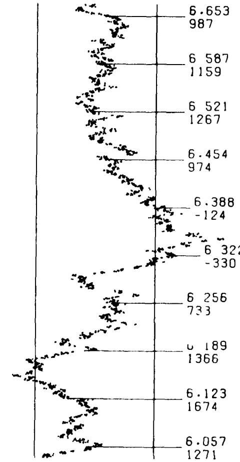

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Fig. 6

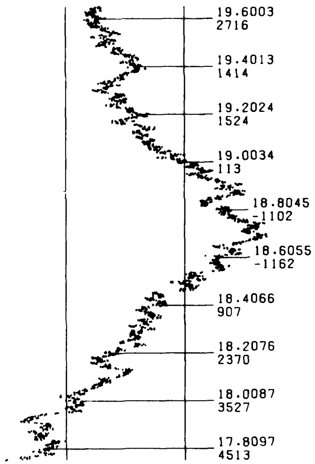


Fig. 7

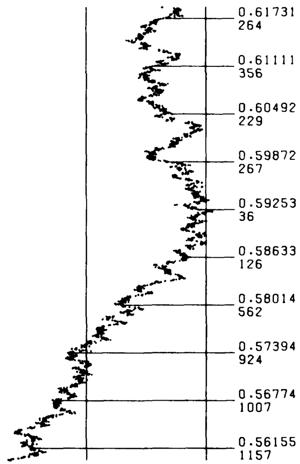


Fig. 8

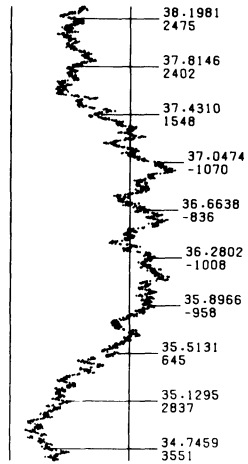


Fig. 9

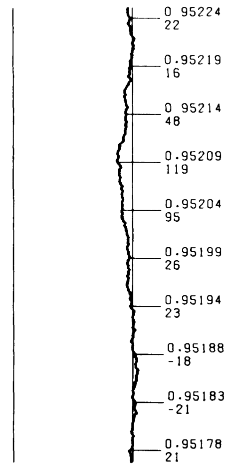
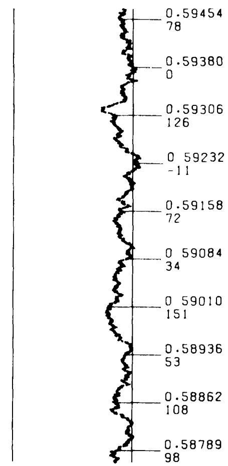


Fig. 10



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