ON THE ALEXANDER POLYNOMIALS OF ALTERNATING TWO-COMPONENT LINKS

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ABSTRACT. Let L be an alternating two-component link with Alexander polynomial

$\Delta(x,y)$. Then the polynomials $(1 - x) \Delta (x,y)$ and $(1 - y) \Delta (x,y)$ are

alternating. That is, $(1 - y) \Delta (x,y)$ can be written as $\sum_{i,j} c_{ij} x^i y^j$ in such

a way that $(-1)^{i+j} c_{ij} \geq 0$.

1. INTRODUCTION.

   This paper extends the graphical methods of Crowell [1] from the study of

the reduced Alexander polynomial $\Delta(t)$ to the study of the unreduced Alexander

polynomial $\Delta(x,y)$. This extension requires extra care in labelling the edges

of our graph and in comparing its adjacency matrix to the Alexander matrix of the

alternating link.

Crowell also used his method to prove that for alternating links,

$\deg \Delta(t) = 2h$, where h is the genus. In a future paper, we hope to prove

equalities relating the x- and y-degrees of $\Delta(x,y)$ to geometric properties

of an alternating link.

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2. ALTERNATING LINK PROJECTIONS.

Let P be an alternating, regular, planar projection of a two-component

link $L = K_1 \cup K_2$, such as the projection of the link 6_3^2 [4, p.416] in Fig. 1.

The components of L are oriented, and P inherits this link-orientation.

The two thick arrows off the projection in Fig. 1 indicate link orientation.

We call K_1 the x-component and K_2 the y-component.

The crossings of a projection are either positive, as shown in Fig. 2a), or

negative, as shown in Fig. 2b), depending upon the orientations of the constitu-

ent segments. We also distinguish four types of crossings depending upon how

the x- and y-components enter in. These are:

> r_x- crossings : x-component overcrosses y-component.
>
> r_y- crossings : y-component overcrosses x-component.
>
> s_x- crossings : x-component overcrosses itself.
>
> s_y- crossings : y-component overcrosses itself.

The capital letters R_x, R_y, S_x, and S_y denote the number of crossings of each

type. We follow Crowell's convention that if $R_x = R_y = 0$ for a projection P,

then P is not alternating.

We shall need the following not-quite-obvious fact:

LEMMA 2.1. In an alternating link projection P, $R_x = R_y$.

PROOF. Let O_x be the union of the r_x- and s_x-crossings and let U_x be the union of the r_y- and s_x-crossings. Let (i) be a crossing in O_x. The overcrossing x-component at (i) must terminate (go under) at an adjacent vertex (j) ϵ U_x, since the projection is alternating. Define $f((i)) = (j)$. f is a one-to-one correspondence between O_x and U_x, so $R_x + S_x = R_y + S_x$ and $R_x = R_y$.   □

If a projection has d crossings labelled (1),(2),...,(d), we stipulate that (1) be an r_x-crossing, (2),...,($2R_x$) be r_x- or r_y-crossings, and ($2R_x+1$),...,(d) be s_x- or s_y-crossings.

A <u>Wirtinger</u> <u>presentation</u> [2, p.86] for the link group $\Pi_1(S^3 - L)$ has one generator for each overcrossing segment in a regular projection of the link. If the projection is alternating, we can label the segment which overcrosses at vertex (i) "x_i" or "y_i", depending on whether it belongs to the x- or y-component of the link. (Thus we define only one of the two symbols "x_i", "y_i".) The relators r_i, i = 1,...,d, of the Wirtinger presentation are of the form

$$r_i = x_j \, y_i \, x_k^{-1} \, y_i^{-1}$$

for the crossing of Fig. 2a) and of the form

$$r_i = x_i \, y_j \, x_i^{-1} \, y_k^{-1}$$

for the crossing of Fig. 2b).

3. <u>ASSOCIATED GRAPH THEORY</u>.

We now regard the projection P as a graph with the crossings of P as vertices and the segments of P joining vertices as edges. Thus each over-crossing segment in an alternating projection contributes two edges to the graph. We will use the word "vertex" when we are thinking of P strictly as

a graph, and the word "crossing" when we are thinking of P as a link projection.

We next orient and label the edges of the graph of P. This <u>alternating</u>

<u>orientation</u> differs from the link orientation. At each vertex (i) of P, we

orient the two incident edges which belong to undercrossing segments at (i)

<u>toward</u> (i). The edge which is on the left with respect to the overcrossing seg-

ment receives the label "1". The edge on the right receives the label "-x"

if the overcrossing segment belongs to the x-component and "-y" if the over-

crossing segment belongs to the y-component. Fig. 1 shows the alternating

orientation and labelling of 6^2_3 . We indicate the alternating orientation by

arrows <u>on</u> the edges of P. We differ from Crowell [1, pp.260-61] only in using

"-x" and "-y" where he uses simply "-t".

The <u>adjacency</u> <u>matrix</u> $B = (b_{ij})$ of an oriented and labelled graph is the

$d \times d$ matrix having for (i,j) entry, $i \neq j$, the sum of the labels on the

edges which start at j and end at i. (If there are no such edges, then

$b_{ij} = 0$.) We also define $b_{ii} = - \sum_{j \neq i} b_{ij}$. Thus each row-sum of B is zero.

THEOREM 3.1 (Matrix-tree theorem). Let B' be the matrix obtained from an

adjacency matrix B by crossing out the first row and column. Then

$\det B' = \sum_T \Pi(T)$, where the summation is over all maximal rooted trees T with

origin (1) in the graph [1, footnote 2, p. 262] and $\Pi(T)$ denotes the product

of the labels on the edges of T.

PROOF. See [3, p.47]. Our matrix B is the transpose of Crowell's (dis-

regarding the difference in labelling). Crowell makes his column-sums zero

while Moon [3] makes his row-sums zero. This discrepancy arises because Crowell

is counting trees oriented <u>away</u> <u>from</u> the origin (as we are) while Moon is counting

trees oriented <u>toward</u> the origin.

4. <u>COMPARISON WITH THE ALEXANDER MATRIX</u>.

Our next task is to prove that the polynomial det B' given in the last
theorem is a link-type invariant.

The <u>Alexander matrix</u> A [2, p.100] of L with respect to the projection P
and its Wirtinger presentation is the $d \times d$ matrix $(a_{ij}) = \left(\alpha\gamma\left(\frac{\partial r_i}{\partial x_j}\right)\right)$. (The

map $\gamma : F_d \to \Pi_1(S^3 - L)$ displays the link group as a homomorphic image of a free

group, and the Abelianizing map α sends all $x_i \to x$ and all $y_i \to y$.) The rows

and columns of A correspond naturally to the vertices (1),...,(d) of P since

the relators and generators of the Wirtinger presentation so correspond. Define

$\overline{A}$ to be the matrix obtained from A by multiplying any row which corresponds

to a negative crossing by -1. From now on, we will call $\overline{A}$ the Alexander

matrix.

LEMMA 4.1. Let $\overline{A}'$ be the matrix obtained from $\overline{A}$ by deleting the first

row and column. Then there is a polynomial $\Delta(x,y)$ such that, up to units

$\pm x^{\pm p} y^{\pm q}$ in $\mathbb{Z}[x, x^{-1}, y, y^{-1}]$, det $\overline{A}' = (1-x)\Delta(x,y)$.

PROOF. This is a special case of a theorem of Torres [5, p.61].   □

$\Delta(x,y)$ is defined to be the <u>Alexander polynomial</u> of the link. It is a

link-type invariant [2, p.120]. If we had deleted a column of $\overline{A}$ corresponding

to a segment from the y-component, we would have gotten det $\overline{A}' = (1-y)\Delta(x,y)$.

THEOREM 4.2. For a given alternating link projection, the adjacency matrix B

and the Alexander matrix $\overline{A}$ are identical, except for the diagonal entries in

rows corresponding to r_x- or r_y-crossings. If (i) is an r_x- (resp. r_y-)

crossing, then $b_{ii} = x-1$ (resp. y-1) while $a_{ii} = y-1$ (resp. x-1).

PROOF. This is a routine application of the free differential calculus, and

exactly follows [1, (2.10), p.261]. Notice that b_{ii} is defined to be minus the

sum of the labels on all edges coming <u>in</u> to vertex (i). This sum must be either

x-1 or y-1.   □

We give below the matrices $\overline{A}$ and B for the alternating projection of the

link 6^2_3 shown in Fig. 1.

$$
\begin{array}{c}
\begin{array}{ccccccc}
 & (1) & (2) & (3) & (4) & (5) & (6) \\
\end{array} \\
\begin{array}{c}
(1) \\
(2) \\
(3) \\
(4) \\
(5) \\
(6)
\end{array}
\left|
\begin{array}{cccccc}
x-1 & -x & 0 & 0 & 1 & 0 \\
1 & y-1 & 0 & -y & 0 & 0 \\
-y & 0 & y-1 & 1 & 0 & 0 \\
0 & 0 & -x & x-1 & 0 & 1 \\
0 & 1 & 0 & 0 & y-1 & -y \\
0 & 0 & 1 & 0 & -y & y-1
\end{array}
\right.
\left(
\begin{array}{cccccc}
y-1 & -x & 0 & 0 & 1 & 0 \\
1 & x-1 & 0 & -y & 0 & 0 \\
-y & 0 & x-1 & 1 & 0 & 0 \\
0 & 0 & -x & y-1 & 0 & 1 \\
0 & 1 & 0 & 0 & y-1 & -y \\
0 & 0 & 1 & 0 & -y & y-1
\end{array}
\right)
\end{array}
$$

$$\qquad\qquad B \qquad\qquad\qquad\qquad\qquad \overline{A}$$

Recall that $\det B' = \sum\limits_{\sigma\in\Sigma} (-1)^\sigma \prod\limits_{i=2}^{d} b_{i\sigma(i)}$, where Σ is the group of

permutations of $\{2,3,\ldots,d\}$. Let $b_\sigma = \prod\limits_{i=2}^{d} b_{i\sigma(i)}$. Any permutation σ can be

written as a product of (algebraic) cycles $(i,\sigma(i), \sigma(\sigma(i)),\ldots)(j,\sigma(j),$

$\sigma(\sigma(j)),\ldots)\ldots$. A (geometric) cycle in a graph is a collection of oriented

edges which forms a closed curve.

LEMMA 4.3. Let b_σ be a non-zero entry in $\det B'$. Then for every non-

trivial cycle $(i,\sigma(i), \ldots)$ in σ, there is a cycle of edges in P joining

vertex (i) to vertex $\sigma(i)$, etc.

PROOF. This is clear from the definition of the adjacency matrix.   □

In Fig. 1, hashmarks indicate the (geometric) cycles corresponding to

$\sigma = (3\ 4)(5\ 6)$. For this example, $b_\sigma = (y-1)(1)(-x)(-y)(-y)$.

LEMMA 4.4. Suppose there is an edge of the graph of P joining vertex (i)

to vertex $\sigma(i)$. Then if (i) is an r_x-crossing, $\sigma(i)$ is an r_y- or s_x-

crossing. If (i) is an r_y-crossing, then $\sigma(i)$ is an r_x- or s_y-crossing.

PROOF. The edge in question is part of an overcrossing segment at (i)

which belongs to the x- (resp. y-) component. At $\sigma(i)$, this overcrossing

segment must terminate. Thus $\sigma(i)$ is an r_y- or s_x- (resp. r_x- or s_y-)

crossing.   □

COROLLARY 4.5. If $b_o \neq 0$, the number of off-diagonal factors b_{io} coming from r_x-rows equals the number of off-diagonal elements coming from r_y-rows. $\square$

COROLLARY 4.6. If $b_o \neq 0$, then there is a positive integer p_1 and non-negative integers p_2, p_3, p_4, p_5 such that

$$b_o = (x-1)^{p_1-1}(y-1)^{p_1}(-x)^{p_2}(-y)^{p_3}(x-1)^{p_4}(y-1)^{p_5}.$$

The corresponding term of $\det \overline{A}'$ is

$$a_o = (x-1)^{p_1}(y-1)^{p_1-1}(-x)^{p_2}(-y)^{p_3}(x-1)^{p_4}(y-1)^{p_5}.$$

PROOF. In B', there are $R_x - 1$ r_x-rows and R_x r_y-rows. Since the off-diagonal factors of b_o must be paired between r_x- and r_y-rows, there must be an imbalance of 1 in the diagonal elements. In a_o, this imbalance is reversed. $\square$

THEOREM 4.7. $\det B' = (1-y)\Delta(x,y)$, up to units in $\mathbb{Z}[x, x^{-1}, y, y^{-1}]$.

PROOF. By Lemma 4.1, $\det \overline{A}' = \sum_{o \in \Sigma} (-1)^o a_o = (1-x)\Delta(x,y)$, up to units. Let $a_o = (x-1)a_o'$. By Corollary 4.6, $b_o = (y-1)a_o'$. Thus $\det B' =$

$$= \sum_o (-1)^o b_o = (1-y)\Delta(x,y), \text{ up to units.}$$
$\square$

THEOREM 4.8. The polynomials $(1-y)\Delta(x,y)$ and $(1-x)\Delta(x,y)$ are alternating. That is, $(1-y)\Delta(x,y)$ can be written as $\sum_{i,j} c_{ij} x^i y^j$ in such a way that $(-1)^{i+j} c_{ij} \geq 0$.

PROOF. The argument is identical to [1, (2.13), p.262]. By Theorem 3.1, we can compute $\det B' = (1-y)\Delta(x,y)$ by summing the products of the labels on maximal rooted trees in the graph of P. One of these products is positive if and only if it has an even number of edges labelled "-x" or "-y". All the results we have developed for $(1-y)\Delta(x,y)$ can be duplicated for $(1-x)\Delta(x,y)$ by making vertex (1) an r_y-crossing. $\square$

It would be interesting to know whether the polynomial $\Delta(x,y)$ must be alternating for alternating links. This does not follow by elementary algebra from our results. For example, the polynomial

$$3 - 2x + 2x^2$$
$$-2y - xy - 2x^2 y$$
$$+2y^2 - 2xy^2 + 3x^2 y^2$$

becomes alternating upon multiplication by $(1-x)$ or $(1-y)$. According to the Torres conditions [5, p.57], this polynomial could be the Alexander polynomial of a linking-number-one link each of whose components has Alexander polynomial $3 - 5t + 3t^2$.

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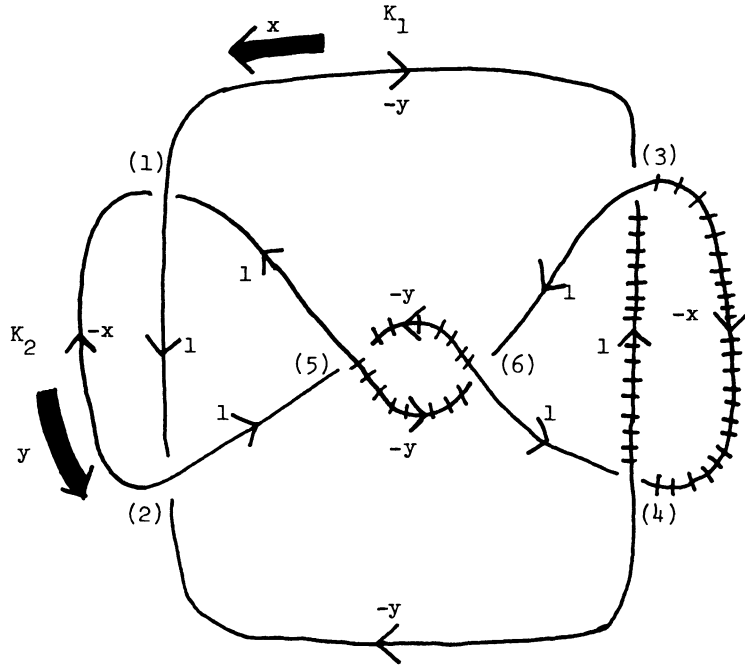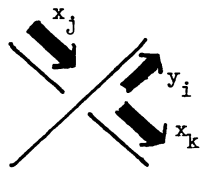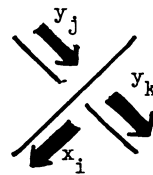5. Torres, G. On the Alexander polynomial, Annals of Math. 57 (1953) 57-89.

FIGURE 1

FIGURE 2