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AN $n\times n$ MATRIX OF LINEAR FUNCTIONALS OF C^*-ALGEBRAS

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We show that any bounded matrix of linear functionals $[f_{ij}] : M_n(A) \to M_n(\mathbb{C})$ has a representation $f_{ij}(a) = \langle T\pi(a)x_j, x_i\rangle$, $a \in A$, $i,j = 1,2,\ldots,n$, for some representation π on a Hilbert space K and an n vectors $x_1, x_2, \ldots, x_n$ in K.

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1. Introduction. Let M_n be the C^*-algebras of complex $n\times n$ matrices generated as a linear space by the matrix units $E_{ij}(i,j = 1,2,\ldots,n)$ and let $B(H)$ denotes the algebra of all bounded linear operators on a Hilbert space H. Let A and B denote C^*-algebras and $L : A \to B$ be a bounded linear map. The map L is positive provided $L(a)$ is positive whenever a is positive. The map L is said to be completely positive if $L\otimes I_n : A\otimes M_n \to B\otimes M_n$ defined by $L\otimes I_n(a\otimes b) = L(a)\otimes b$ is positive for all n. The map L is said to be completely bounded if $\sup_n \|L\otimes I_n\|$ is finite. We set $\|L\|_{cb} = \sup_n \|L\otimes I_n\|$, $L^*(a) = L(a)^*$. Given $S \subseteq B(H)$, and let S' denote its commutant. An $n\times n$ matrix $[f_{ij}]$ of linear functionals on a C^*-algebra A is positive if $[f_{ij}(a_{ij})]$ is positive whenever $[a_{ij}]$ is positive in $A\otimes M_n$.

2. A positive matrix of linear functionals. The following result [7, Corollary 2.3] is well known.

Theorem 2.1. *Let F be a linear map from a C^*-algebra A to M_n and let the functional $f : A\otimes M_n \to C$ be defined by $f(a\otimes E_{ij}) = [F(a)]_{ij}$. If f is positive, then F is completely positive.*

Depending on the previous result, Suen [8] proved the following theorem.

Theorem 2.2. *Let $F = [f_{ij}] : A\otimes M_n \to M_n(\mathbb{C})$ be a positive $n\times n$ matrix of linear functionals on A, then F is completely positive.*

In what follows we give a new proof to this result.

Proof. Define $L : (M_n(A))\otimes M_n \to C$ by

$$L([a_{kl}]\otimes E_{ij}) = (F[a_{kl}])_{ij} = f_{ij}(a_{ij}), \tag{2.1}$$

and a complete positive map $\delta : M_n(A) \to A$ by $\delta[a_{ij}] = \sum_{i,j} a_{ij}$ and put

$$E = \begin{pmatrix} E_{11} & & 0 \\ & \ddots & \\ 0 & & E_{nn} \end{pmatrix}. \tag{2.2}$$

Let $[a_{kl}^{ij}]_{ijkl}$ be a positive element in $M_n(A)\otimes M_n$ we have

$$L\big[a_{kl}^{ij}\big]_{ijkl} = L\Big(\sum_{ij}[a_{kl}]\otimes E_{ij}\Big) = \sum_{ij} L([a_{kl}]\otimes E_{ij}) \\ = \sum_{ij} f_{ij}(a_{ij}) = \delta\circ F[a_{ij}] \geq 0, \tag{2.3}$$

as $[a_{ij}] \equiv [a_{ij}^{ij}]$ is positive via its identification with $E[a_{kl}^{ij}]_{ijkl}E$ which is positive. Another method, let

$$\Phi = \delta\circ F : M_n(A) \longrightarrow M_n(\mathbb{C}) \longrightarrow \mathbb{C}. \tag{2.4}$$

As F, δ are positive maps, then Φ is positive. Since $\mathbb{C}$ is commutative, then by [2] Φ is completely positive. The complete positivity of Φ and δ insures the complete positivity of F. □

Choi [2] showed that any n-positive map from a C^*-algebra A to M_n is completely positive. The following is a generalization of a special case.

Theorem 2.3. *Via the linear functionals $F = [f_{ij}] : M_n(A) \to M_n(\mathbb{C})$, any positive map $\Psi : A \to M_n(\mathbb{C})$ is completely positive.*

Proof. Define a map $\gamma : A \to M_n(A)$ by

$$\gamma(a) = \begin{pmatrix} a & \cdots & a \\ \vdots & \ddots & \vdots \\ a & \cdots & a \end{pmatrix}, \tag{2.5}$$

then γ is completely positive. Write $\Psi = F\circ\gamma : A \to M_n(\mathbb{C})$. The positivity of Ψ and γ insures the positivity of F, in fact $F = \Psi\circ\gamma^{-1}$, and

$$\gamma^{-1} = \frac{1}{n^2}\delta\Big|_{\begin{pmatrix} a & \cdots & a \\ \vdots & \ddots & \vdots \\ a & \cdots & a \end{pmatrix}}. \tag{2.6}$$

Therefore, F is completely positive by Theorem 2.2, which in return gives that Ψ is completely positive. □

Lemma 2.4. (a) *(See [3].) Let $R, S, T \in B(H)$ with T being positive and invertible. Then*

$$\begin{pmatrix} T & S \\ S^* & R \end{pmatrix} \geq 0 \Longleftrightarrow R \geq S^*T^{-1}S. \tag{2.7}$$

(b) *Let $T \in B(H)$, then*

$$\begin{pmatrix} I & S \\ T^* & I \end{pmatrix} \geq 0 \Longleftrightarrow \|T\| \leq 1. \tag{2.8}$$

Proof. (a) This follows from the identity

$$\left\langle \begin{pmatrix} T & S \\ S^* & R \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle = \left\|T^{1/2}x + T^{-1/2}Sy\right\|^2 + \langle (R - S^*T^{-1}S)y, y\rangle \tag{2.9}$$

as

$$\begin{aligned} R-S^*T^{-1}S\ge 0 &\Longrightarrow \left\|T^{1/2}x+T^{-1/2}Sy\right\|^2+\langle (R-S^*T^{-1}S)y,y\rangle\ge 0\\ &\Longrightarrow \begin{pmatrix} T & S\\ S^* & R\end{pmatrix}\ge 0, \end{aligned}\tag{2.10}$$

and if

$$\begin{pmatrix} T & S\\ S^* & R\end{pmatrix}\ge 0,\tag{2.11}$$

choose $T^{1/2}x+T^{-1/2}Sy=0$ which gives that $\langle (R-S^*T^{-1}S)y,y\rangle\ge 0$, that is, $R\ge S^*T^{-1}S$.

(b) Follows from the following two identities:

$$\begin{aligned} \left\langle \begin{pmatrix} I & T\\ T^* & I\end{pmatrix}\begin{pmatrix} x\\ y\end{pmatrix},\begin{pmatrix} x\\ y\end{pmatrix}\right\rangle &= \left\|x+Ty\right\|^2+\|y\|^2-\|Ty\|^2,\\ \left\langle \begin{pmatrix} I & T\\ T^* & I\end{pmatrix}\begin{pmatrix} -Tx\\ x\end{pmatrix},\begin{pmatrix} -Tx\\ x\end{pmatrix}\right\rangle &= \|x\|^2-\|Tx\|^2. \end{aligned}\tag{2.12}$$

□

Theorem 2.5. *Let $F:M_n(A)\to M_n(\mathbb{C})$. If F is bounded then it is completely bounded.*

Proof. Without loss of generality, assume that $\|F\|\le 1$. Therefore, by Lemma 2.4(b),

$$\begin{pmatrix} I_n & F\\ F^* & I_n\end{pmatrix}\ge 0,\tag{2.13}$$

this also follows from Lemma 2.4(a) by noticing that $\|F\|\le 1\Rightarrow \|F\|^2\le 1\Rightarrow \|F^*F\|\le 1\Rightarrow F^*F\le I_n\Rightarrow \left(\begin{smallmatrix} I_n & F\\ F^* & I_n\end{smallmatrix}\right)\ge 0$. Let $\Phi=[\phi_{ij}]:M_n(A)\to M_n(\mathbb{C})$ be defined by

$$\phi_{ij}=\begin{cases} 0, & i\ne j,\\ \alpha\|a\|, & i=j,\ \alpha>0 \text{ is large enough.}\end{cases}\tag{2.14}$$

Clearly, $\Phi-I_n\ge 0$, so that

$$\begin{pmatrix} \Phi-I_n & 0\\ 0 & \Phi-I_n\end{pmatrix}\ge 0,\tag{2.15}$$

which implies that

$$\begin{pmatrix} \Phi-I_n & 0\\ 0 & \Phi-I_n\end{pmatrix}+\begin{pmatrix} I_n & F\\ F^* & I_n\end{pmatrix}=\begin{pmatrix} \Phi & F\\ F^* & \Phi\end{pmatrix}\ge 0.\tag{2.16}$$

By Theorem 2.3,

$$\begin{pmatrix} \Phi & F \\ F^* & \Phi \end{pmatrix} : M_{2n}(A) \longrightarrow M_{2n}(\mathbb{C}) \tag{2.17}$$

is completely positive and hence completely bounded. Therefore F is completely bounded. □

Theorem 2.6. *Let $G : A \to M_n(\mathbb{C})$ be a bounded map defined by $G(a) = [g_{ij}(a)]_{ij}$. Then there is a representation π of A, a Hilbert space K, an isometry $V : H \to K$, and an operator $U_{ij} \in \pi(A)'$ such that $[\pi(a)VH]$ is dense in K and $g_{ij}(\cdot) = V^* U_{ij} \pi(\cdot) V$ with $\|U_{ij}\| \le 2$.*

Proof. Since G is bounded, then by [5, Lemma 6] G is completely bounded. By [6, Theorem 2.5] there exist completely positive maps $\phi = [\phi_{ij}]$, $\varphi = [\varphi_{ij}] : A \to M_n(A)$ such that the map $\Psi : M_2(A) \to M_{2n}(\mathbb{C})$, defined by

$$\Psi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \phi(a) & G(b) \\ G^*(c) & \varphi(d) \end{pmatrix}, \tag{2.18}$$

is completely positive. Define matrices $M_{ij} \in M_{2n}(\mathbb{C})$ by

$$M_{ij} = [r_{kl}] : r_{ij} = \begin{cases} 1, & i = j, \\ 0, & \text{otherwise}. \end{cases} \tag{2.19}$$

The map

$$\begin{pmatrix} \phi_{ii} & f_{ij} \\ f_{ji} & \varphi_{jj} \end{pmatrix}, \quad f_{ji} = f_{ij}^* \tag{2.20}$$

is completely positive, as it is identified with the map $M_{ij} \Psi M_{ij}$, which is completely positive as

$$(M_{ij} \Psi M_{ij}) \otimes M_r = \sum_{k,l=1}^{r} (M_{ij} \Psi M_{ij}) \otimes E_{kl} = M_{ij} \left(\sum_{k,l=1}^{r} \Psi \otimes E_{kl} \right) M_{ij} \ge 0. \tag{2.21}$$

Therefore,

$$\begin{pmatrix} 1 \\ \lambda \end{pmatrix}^* \begin{pmatrix} \phi_{ii} & f_{ij} \\ f_{ji} & \varphi_{jj} \end{pmatrix} \begin{pmatrix} 1 \\ \lambda \end{pmatrix} = \phi_{ii} + \varphi_{jj} + \lambda f_{ij} + \lambda^* f_{ji} \tag{2.22}$$

is completely positive. By setting $\Phi_{ij} = (\phi_{ii} + \varphi_{jj})/2$, we have for any λ for which $|\lambda| = 1$, $\Phi_{ij} + \mathrm{Re}(\lambda f_{ij})$ is completely positive. In particular, $\Phi_{ij} \pm \mathrm{Re}(f_{ij})$ and $\Phi_{ij} \pm \mathrm{Im}(f_{ij})$ are completely positive. If $\Phi = \sum_{i=1}^{n} \Phi_{ii}$, $\Phi \ge \Phi_{ij}$, and the maps $\Phi \pm \mathrm{Re}(f_{ij})$ and $\Phi \pm \mathrm{Im}(f_{ij})$ are completely positive. Let (π, V, K) be the minimal Stinespring representation of Φ, that is, K is a Hilbert space, $V : H \to K$ is an isometry, $\pi : A \to B(K)$ is a unital $*$-representation with $[\pi(A)VH]$ dense in K and $\Phi(a) = V^* \pi(a) V$. Since $\Phi - (\Phi + \mathrm{Re}(f_{ij}))/2$ is completely positive, that is, $\Phi \ge (\Phi + \mathrm{Re}(f_{ij}))/2$ by [1, Theorem 1.4.2], then there exists a unique positive Q_{ij} in $\pi(A)'$, $Q_{ij} \le I$ such that

$V^*Q_{ij}\pi V = (V^*\pi V + \mathrm{Re}(f_{ij}))/2$. Therefore, $\mathrm{Re}(f_{ij}) = V^*(2Q_{ij}-I)\pi V$. Also $\mathrm{Im}(f_{ij}) = V^*(2R_{ij}-I)\pi V$, for a unique positive $R_{ij} \in \pi(A)'$, $R_{ij} \le I$. Write $S_{ij} = 2Q_{ij}-I$, $T_{ij} = 2R_{ij}-I$, $U_{ij} = S_{ij}+iT_{ij}$, $S_{ij} = S_{ij}^*$, $T_{ij} = T_{ij}^*$, $\|S_{ij}\| \le 1$, $\|T_{ij}\| \le 1$, we have $f_{ij} = V^*U_{ij}\pi V$, $U_{ij} \in \pi(A)'$, $\|U_{ij}\| \le 2$. □

The following theorem generalizes [4, Proposition 2.4].

Theorem 2.7. *Let $F = [f_{ij}] : M_n(A) \to M_n(\mathbb{C})$ be bounded. Then there is a representation π of A on a Hilbert space K and n vectors $x_1, x_2, \dots, x_n$ in K, an operator $T \in \pi(A)'$, $\|T\| \le 2$ such that $f_{ij}(a) = \langle T\pi(a)x_j, x_i\rangle$, $a \in A$, $i,j = 1,2,\dots,n$.*

Proof. By [8, Theorem 2.2], F is completely bounded, and by [6, Theorem 2.5] there exist completely positive maps $\phi = [\phi_{ij}]$ and $\varphi = [\varphi_{ij}] : M_n(A) \to M_n(\mathbb{C})$ such that the map

$$\Psi = \begin{pmatrix} \phi & F \\ F^* & \varphi \end{pmatrix} : M_{2n}(A) \longrightarrow M_{2n}(\mathbb{C}) \tag{2.23}$$

is completely positive. For $|\lambda| = 1$, the map

$$\begin{pmatrix} I_n \\ \lambda I_n \end{pmatrix}^* \Psi \begin{pmatrix} B & B \\ B & B \end{pmatrix} \begin{pmatrix} I_n \\ \lambda I_n \end{pmatrix} = \phi(B) + \varphi(B) + \lambda F(B) + (\lambda F)^*(B), \tag{2.24}$$

$B \in M_n(A)$, is completely positive. By setting $\Phi = \phi + \varphi = [\Phi_{ij}]$, the maps $\Phi \pm \mathrm{Re}(F)$ and $\Phi \pm \mathrm{Im}(F)$ are completely positive. Since $\Phi \ge (\Phi + \mathrm{Re}(F))/2$, then by [4, Theorem 2.1] let π be the representation engendered by Φ on a Hilbert space K such that $\Phi_{ij}(a) = \langle \pi(a)x_j, x_i\rangle$, for some generating set of vectors $x_1, x_2, \dots, x_n$ for $\pi(A)$. By [4, Proposition 2.4], there is a positive operator H in the unit ball of $\pi(A)'$ such that $(\Phi + \mathrm{Re}(F))/2 = [\langle H\pi(\cdot)x_j, x_i\rangle]_{ij}$ with

$$\mathrm{Re}(F) = 2[\langle H\pi(\cdot)x_j, x_i\rangle]_{ij} - [\langle \pi(\cdot)x_j, x_i\rangle] = [\langle (2H-I)\pi(\cdot)x_j, x_i\rangle]. \tag{2.25}$$

Let $R = 2H - I$, then $R \in \pi(A)'$, $R = R^*$, $\|R\| \le I$, and $\mathrm{Re}(F) = [\langle S\pi(\cdot)x_j, x_i\rangle]$. Similarly, there exists $R \in \pi(A)'$, $R = R^*$, $\|R\| \le I$ such that $\mathrm{Im}(F) = [\langle R\pi(\cdot)x_j, x_i\rangle]$. Write $T = S + iR$, we have $F(\cdot) = [\langle T\pi(\cdot)x_j, x_i\rangle]$. Therefore, $f_{ij}(a) = \langle T\pi(a)x_j, x_i\rangle$, $T \in \pi(A)'$, $\|T\| \le 2$. □

The following is a generalization of [8, Proposition 2.7].

Theorem 2.8. *If the map $[f_{ij}] : A \otimes M_n \to B(H) \otimes M_n$, defined by $[f_{ij}]([a_{ij}]) = [f_{ij}(a_{ij})]$, is completely bounded, then there is a representation π of A on a Hilbert space K, an isometry $V : H \to K$, and an operator $T_{ij} \in \pi(A)'$ such that $[\pi(A)VH]$ is dense in K and $f_{ij}(\cdot) = V^*T_{ij}\pi(\cdot)V$ with $\|T_{ij}\| \le 2$.*

Proof. The proof it follows by the same technique used in the proof of Theorem 2.6. □

The following generalizes [7, Proposition 4.2] for a special case.

Theorem 2.9. *Via all linear functionals $F = [f_{ij}] : M_n(A) \to M_n(\mathbb{C})$, any positive map $\phi : M_n(\mathbb{C}) \to M_p(\mathbb{C})$ is completely positive.*

Proof. By the following diagram

$$A \xrightarrow{\gamma} M_n(A) \xrightarrow{F} M_n(\mathbb{C}) \xrightarrow{\phi} M_n(p), \tag{2.26}$$

$\Psi = \phi \circ F \circ \gamma : A \to M_n(p)$. The positivity of ϕ, F, and γ implies the positivity of Ψ. By Theorem 2.3, Ψ is completely positive. The complete positivity of Ψ, F, and γ insures the complete positivity of ϕ. □

Theorem 2.10. *There is a one-to-one correspondence between the set of all bounded linear functionals $f = [f_{ij}]$ of a C^*-algebra A and the set of all bounded maps $F : A \to M_n(\mathbb{C})$ given by $F_f(a) = [f_{ij}(a)]$.*

Proof. The map f is completely bounded, by [8, Theorem 2.2]. By [6, Theorem 2.5], there exist completely positive maps $\phi, \varphi : M_n(A) \to M_n(\mathbb{C})$ defined by $\phi[a_{ij}] = [\phi_{ij}(a_{ij})]$ and $\varphi[a_{ij}] = [\varphi_{ij}(a_{ij})]$ such that the map $\Phi : M_{2n}(A) \to M_{2n}(\mathbb{C})$, defined by

$$\Phi \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} = \begin{pmatrix} \phi(B_1) & F(B_2) \\ F^*(B_3) & \varphi(B_4) \end{pmatrix}, \quad B_i \in M_n(A), \tag{2.27}$$

is completely positive. If we set $\Phi_{ij} = \phi_{ij}$, $f_{ij} = \Phi_{i,j+n}$, $\varphi_{ij} = \Phi_{i+n,j+n}$, $i, j = 1, 2, \ldots, n$, we have $\Phi = [\Phi_{kl}]$, $k, l = 1, 2, \ldots, 2n$. The map $\Psi_\Phi : M_2(A) \to M_{2n}(\mathbb{C})$, defined by

$$\Psi_\Phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} [\phi_{ij}(a)] & [f_{ij}(b)] \\ [f^*_{ji}(c)] & [\varphi_{ij}(d)] \end{pmatrix}, \tag{2.28}$$

is positive as

$$\Psi_\Phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \Phi \left(E\gamma \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad E^* \right), \tag{2.29}$$

where $\gamma : M_2(A) \to M_{2n}(A)$ is defined by

$$\gamma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes M_n,$$
$$E_{2n\times 2n} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}. \tag{2.30}$$

By [8, Theorem 2.2], Ψ_Φ is completely positive. By [4, Proposition 2.6], there is a one-to-one correspondence between Ψ_Φ and Φ. By putting $a = c = d = 0$, we obtain a one-to-one correspondence between F_f and F. □

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