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COMMON FIXED POINT THEOREMS FOR A WEAK DISTANCE IN COMPLETE METRIC SPACES

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Using the concept of a w-distance, we obtain common fixed point theorems on complete metric spaces. Our results generalize the corresponding theorems of Jungck, Fisher, Dien, and Liu.

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1. Introduction. In 1976, Caristi [1] proved a fixed point theorem in a complete metric space which generalizes the Banach contraction principle. This theorem is very useful and has many applications. Later, Dien [3] showed that a pair of mappings satisfying both the Banach contraction principle and Caristi's condition in a complete metric space has a common fixed point. That is to say, let (X,d) be a complete metric space and let S and T be two orbitally continuous mappings of X into itself. Suppose that there exists a finite number of functions $\{\varphi_i\}_{1\le i\le N_0}$ of X into $\mathbb{R}_+$ such that

$$d(Sx,Ty) \le q\cdot d(x,y)+\sum_{i=1}^{N_0}\left[\varphi_i(x)-\varphi_i(Sx)+\varphi_i(y)-\varphi_i(Ty)\right] \tag{1.1}$$

for all $x,y\in X$ and some $q\in[0,1)$. Then S and T have a unique common fixed point z in X. Further, if $x\in X$ then $S^n x\to z$ and $T^n x\to z$ as $n\to\infty$. In particular, if S is an identity mapping, $q=0$, and $N_0=1$, then this means a Caristi's fixed point theorem.

Recently, Liu [7] obtained necessary and sufficient conditions for the existence of fixed point of continuous self-mapping by using the ideas of Jungck [5] and Dien [3]: let f be a continuous self-mapping of a metric space (X,d), then f has a fixed point in X if and only if there exist $z\in X$, a mapping $g: X\to X$, and a function Φ from X into $[0,\infty)$ such that f and g are compatible, $g(X)\subset f(X)$, g is continuous, and

$$d(gx,z)\le rd(fx,z)+\left[\Phi(fx)-\Phi(gx)\right] \tag{1.2}$$

for all $x\in X$ and some $r\in[0,1)$.

In 1996, Kada et al. [6] introduced the concept of w-distance on a metric space as follows: let X be a metric space with metric d, then a function $p: X\times X\to[0,\infty)$ is called a w-distance on X if the following are satisfied:

(1) $p(x,z)\le p(x,y)+p(y,z)$ for any $x,y,z\in X$;

(2) for any $x\in X$, $p(x,\cdot): X\to[0,\infty)$ is lower semicontinuous;

(3) for any $\epsilon > 0$, there exists $\delta > 0$ such that $p(z,x) \le \delta$ and $p(z,y) \le \delta$ imply $d(x,y) \le \epsilon$.

In this paper, using the concept of a w-distance, we obtain common fixed point theorems on complete metric spaces. Our results generalize the corresponding theorems of Jungck [5], Fisher [4], Dien [3], and Liu [7].

2. Definitions and preliminaries.

Throughout, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}_+$ the set of nonnegative real numbers, that is, $\mathbb{R}_+ := [0,\infty)$.

Definition 2.1 (see [3]). A mapping T of a space X into itself is said to be orbitally continuous if $x_0 \in X$ such that $x_0 = \lim_{i\to\infty} T^{n_i}x$ for some $x \in X$, then $Tx_0 = \lim_{i\to\infty} T(T^{n_i}x)$.

Definition 2.2 (see [2]). Let T be a mapping of a metric space X into itself. For each $x \in X$, let

$$O(T,x,n) = \{x, Tx, \dots, T^n x\}, \quad n = 1,2,\dots,$$
$$O(T,x,\infty) = \{x, Tx, \dots\}. \tag{2.1}$$

A space X is said to be T-orbitally complete if and only if every Cauchy sequence, which is contained in $O(T,x,\infty)$ for some $x \in X$, converges in X.

Definition 2.3 (see [6]). Let X be a metric space with metric d. Then a function $p : X \times X \to \mathbb{R}_+$ is called a w-distance on X if the following properties are satisfied:

(1) $p(x,z) \le p(x,y) + p(y,z)$ for any $x,y,z \in X$;

(2) for any $x \in X$, $p(x,\cdot) : X \to \mathbb{R}_+$ is lower semicontinuous;

(3) for any $\epsilon > 0$, there exists $\delta > 0$ such that $p(z,x) \le \delta$ and $p(z,y) \le \delta$ imply $d(x,y) \le \epsilon$.

The metric d is a w-distance on X. Other examples of w-distance are stated in [6].

Definition 2.4 (see [5]). Let (X,d) be a metric space and $f,g : X \to X$. The mappings f and g are called compatible if and only if for every sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$, it implies

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0. \tag{2.2}$$

Lemma 2.5 (see [6]). *Let X be a metric space with metric d, and p a w-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X, let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $\mathbb{R}_+$ converging to 0, and let $x,y,z \in X$. Then the following properties hold:*

(i) *if $p(x_n,y) \le \alpha_n$ and $p(x_n,z) \le \beta_n$ for any $n \in \mathbb{N}$, then $y = z$. In particular, if $p(x,y) = 0$ and $p(x,z) = 0$, then $y = z$;*

(ii) *if $p(x_n,y_n) \le \alpha_n$ and $p(x_n,z) \le \beta_n$ for any $n \in \mathbb{N}$, then $\{y_n\}$ converges to z;*

(iii) *if $p(x_n,x_m) \le \alpha_n$ for any $n,m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence;*

(iv) *if $p(y,x_n) \le \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.*

3. Main results

Theorem 3.1. *Let* (X,d) *be a complete metric space with a* w*-distance* p*. Suppose that two mappings* $f,g : X \to X$ *and a function* φ *from* X *into* $\mathbb{R}_+$ *are satisfying the following conditions:*

(i) $g(X) \subseteq f(X)$,

(ii) *there exists* $t \in X$ *such that* $p(t,gx) \le r \cdot p(t,fx) + [\varphi(fx) - \varphi(gx)]$ *for all* $x \in X$ *and some* $r \in [0,1)$,

(iii) *for every sequence* $\{x_n\}_{n\in\mathbb{N}}$ *in* X *satisfying*

$$\lim_{n\to\infty} p(t,fx_n) = \lim_{n\to\infty} p(t,gx_n) = 0, \tag{3.1}$$

it implies that

$$\lim_{n\to\infty} \max\{p(t,fx_n), p(t,gx_n), p(fgx_n,gfx_n)\} = 0, \tag{3.2}$$

(iv) *for each* $u \in X$ *with* $u \ne fu$ *or* $u \ne gu$,

$$\inf\{p(u,fx) + p(u,gx) + p(fgx,gfx) : x \in X\} > 0. \tag{3.3}$$

Then f *and* g *have a unique common fixed point in* X.

Proof. Let x_0 be a given point of X. By (i), there exists $x_n \in X$ such that $gx_{n-1} = fx_n$ for $n \ge 1$. From Theorem 3.1(ii), we have

$$p(t,fx_{j+1}) = p(t,gx_j) \le r \cdot p(t,fx_j) + [\varphi(fx_j) - \varphi(gx_j)], \tag{3.4}$$

which implies that

$$\sum_{j=0}^{n-1} p(t,fx_{j+1}) \le r \cdot \sum_{j=0}^{n-1} p(t,fx_j) + \sum_{j=0}^{n-1} [\varphi(fx_j) - \varphi(gx_j)], \tag{3.5}$$

that is,

$$\begin{aligned}\sum_{j=1}^{n} p(t,fx_j) &\le \frac{r}{1-r} p(t,fx_0) + \frac{1}{1-r}[\varphi(fx_0) - \varphi(fx_n)], \\ &\le \frac{r}{1-r} p(t,fx_0) + \frac{1}{1-r}\varphi(fx_0),\end{aligned} \tag{3.6}$$

which means that the series $\sum_{n=1}^{\infty} p(t,fx_n)$ is convergent, so

$$\lim_{n\to\infty} p(t,fx_n) = \lim_{n\to\infty} p(t,gx_n) = 0. \tag{3.7}$$

Suppose that $t \neq ft$ or $t \neq gt$. Then, from Theorem 3.1(iii) and (iv) we obtain that

$$\begin{aligned}0 &< \inf\{p(t,fx)+p(t,gx)+p(fgx,gfx) : x \in X\}\\ &\leq \inf\{p(t,fx_n)+p(t,gx_n)+p(fgx_n,gfx_n) : n \in \mathbb{N}\}\\ &= 0.\end{aligned} \tag{3.8}$$

This is a contradiction. Hence t is a common fixed point of f and g.

We prove that t is a unique common fixed point of f and g. Let u be a common fixed point of f and g. Then, by Theorem 3.1(ii),

$$\begin{aligned}p(t,t) &= p(t,gt) \leq r\cdot p(t,ft)+[\varphi(ft)-\varphi(gt)] = r\cdot p(t,t),\\ p(t,u) &= p(t,gu) \leq r\cdot p(t,fu)+[\varphi(fu)-\varphi(gu)] = r\cdot p(t,u).\end{aligned} \tag{3.9}$$

Thus $p(t,t) = p(t,u) = 0$. From Lemma 2.5, we obtain $t = u$. Therefore t is a unique common fixed point of f and g. □

Remark 3.2. Theorem 3.1 generalizes and improves Dien [3, Theorem 2.2] and Liu [7, Theorem 3.2].

Theorem 3.3. *Let f be a continuous self-mapping of metric space (X,d). Assume that f has a fixed point in X. Then there exists a w-distance $p, t \in X$, a continuous mapping $g : X \to X$, and a function φ from X into $\mathbb{R}_+$ satisfying Theorem 3.1(i), (ii), (iii), and (iv).*

Proof. Let z be a fixed point of f, $r = 1/2$, $gx = t = z$, and $\varphi(x) = 1$ for all $x \in X$. Define $p : X\times X \to \mathbb{R}_+$ by

$$p(x,y) = \max\{d(fx,x),d(fx,y),d(fx,fy)\} \quad \forall x,y \in X. \tag{3.10}$$

Suppose that

$$\lim_{n\to\infty} p(t,fx_n) = \lim_{n\to\infty} p(t,gx_n) = 0. \tag{3.11}$$

Then it is easy to verify that the results of Theorem 3.3 follow. □

Theorem 3.4. *Let f and g be a continuous compatible self-mappings of the metric space (X,d). There exists $t \in X$ satisfying*

$$d(t,gx) \leq r\cdot d(t,fx)+[\varphi(fx)-\varphi(gx)] \tag{3.12}$$

for all $x \in X$ and some $r \in [0,1)$. Then

(i) *for every sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that*

$$\lim_{n\to\infty} d(t,fx_n) = \lim_{n\to\infty} d(t,gx_n) = 0 \tag{3.13}$$

for some $t \in X$, it implies that

$$\lim_{n\to\infty} \max\{d(t,fx_n),d(t,gx_n),d(fgx_n,gfx_n)\} = 0; \tag{3.14}$$

(ii) *for each* $u \in X$ *with* $u \neq fu$ *or* $u \neq gu$,

$$\inf\{d(u,fx)+d(u,gx)+d(fgx,gfx):x\in X\}>0. \tag{3.15}$$

Proof. The results follow by elementary calculation. □

Remark 3.5. Since the metric d is w-distance, from Theorems 3.1, 3.3, and 3.4, we obtain Liu [7, Theorem 3.1].

Theorem 3.6. *Let* (X,d) *be a complete metric space with a* w*-distance* p*, two mappings* $f,g:X\to X$*, and two functions* φ*,* ψ *from* X *into* $\mathbb{R}_+$ *such that Theorem 3.1(i), (iv) are satisfied,*

(i) *for every sequence* $\{x_n\}_{n\in\mathbb{N}}$ *in* X *such that*

$$\lim_{n\to\infty} fx_n=\lim_{n\to\infty} gx_n=t \tag{3.16}$$

for some $t\in X$*, it implies that*

$$\lim_{n\to\infty}\max\{p(t,fx_n),p(t,gx_n),p(fgx_n,gfx_n)\}=0, \tag{3.17}$$

(ii)

$$\begin{aligned}p(gx,gy)\le{}& a_1p(fx,fy)+a_2p(fx,gx)+a_3p(fy,gy)\\&+a_4p(fx,gy)+a_5[p(gx,fy)d(fy,gx)]^{1/2}\\&+[\varphi(fx)-\varphi(gx)]+[\psi(fy)-\psi(gy)]\end{aligned} \tag{3.18}$$

for all $x,y\in X$*, where* a_1*,* a_2*,* a_3*,* a_4*, and* a_5 *are in* $[0,1)$ *with* $a_1+a_4+a_5<1$ *and* $a_1+a_2+a_3+2a_4<1$*.*

Then f *and* g *have a unique common fixed point in* X*.*

Proof. Let x_0 be an arbitrary point of X. By Theorem 3.1(i), we obtain a sequence $\{x_n\}$ in X such that $gx_{n-1}=fx_n$ for $n\ge 1$. Let $y_n=p(fx_n,fx_{n+1})$ for $n\ge 0$. It follows from Theorem 3.6(ii) that

$$\begin{aligned}y_{j+1}&=p(gx_j,gx_{j+1})\\&\le a_1p(fx_j,fx_{j+1})+a_2p(fx_j,gx_j)+a_3p(fx_{j+1},gx_{j+1})\\&\quad+a_4p(fx_j,gx_{j+1})+a_5[p(gx_j,fx_{j+1})d(fx_{j+1},gx_j)]^{1/2}\\&\quad+[\varphi(fx_j)-\varphi(gx_j)]+[\psi(fx_{j+1})-\psi(gx_{j+1})]\\&\le(a_1+a_2+a_4)y_j+(a_3+a_4)y_{j+1}\\&\quad+[\varphi(fx_j)-\varphi(fx_{j+1})]+[\psi(fx_{j+1})-\psi(fx_{j+2})],\end{aligned} \tag{3.19}$$

which implies that

$$y_{j+1}\le L_1y_j+L_2[\varphi(fx_j)-\varphi(fx_{j+1})+\psi(fx_{j+1})-\psi(fx_{j+2})], \tag{3.20}$$

where

$$L_1=\frac{a_1+a_2+a_4}{1-a_3-a_4},\qquad L_2=\frac{1}{1-a_3-a_4}. \tag{3.21}$$

Thus

$$\sum_{j=1}^{n} y_j \leq \frac{L_1}{1-L_1} y_0 + \frac{L_2}{1-L_1} [\varphi(fx_0) + \psi(fx_1)] \tag{3.22}$$

for all $n \geq 1$. Hence, the series $\sum_{n=1}^{\infty} y_n$ is convergent. For any $n, r \geq 1$, we have

$$p(fx_n, fx_{n+r}) \leq \sum_{i=n}^{n+r-1} y_i. \tag{3.23}$$

By Lemma 2.5, this implies that $\{fx_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Since X is a complete metric space, there exists $t \in X$ such that $fx_n \to t$ as $n \to \infty$. From Theorem 3.6(i), we have

$$\lim_{n\to\infty} \max\{p(t, fx_n), p(t, gx_n), p(fgx_n, gfx_n)\} = 0. \tag{3.24}$$

Suppose that $t \neq ft$ or $t \neq gt$, then from Theorem 3.1(iv) we obtain that

$$\begin{aligned} 0 &< \inf\{p(t, fx) + p(t, gx) + p(fgx, gfx) : x \in X\} \\ &\leq \inf\{p(t, fx_n) + p(t, gx_n) + p(fgx_n, gfx_n) : n \in \mathbb{N}\} \\ &= 0. \end{aligned} \tag{3.25}$$

which is a contradiction. Therefore t is a common fixed point of f and g. It follows from Lemma 2.5 and Theorem 3.6(ii) that t is a unique common fixed point of f and g. □

Theorem 3.7. *Let f be a continuous self-mapping of a metric space (X, d). Assume that f has a fixed point in X. Then there exist a w-distance p, $t \in X$, a continuous mapping $g : X \to X$, and functions φ, ψ from X into $\mathbb{R}_+$ satisfying Theorem 3.1(i), (iv) and Theorem 3.6(i), (ii).*

Proof. By a method similar to that in the proof of Theorem 3.3, the results follow. □

Remark 3.8. Since the metric d is w-distance, from Theorems 3.4, 3.6, and 3.7, we obtain Jungck [5, Theorem], Fisher [4, Theorem 2], and Liu [7, Theorem 3.3].

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