IJMMS 30:11 (2002) 697–703
PII. S0161171202013017
http://ijmms.hindawi.com

ESTIMATES FOR THE NORMS OF SOLUTIONS OF DELAY DIFFERENCE SYSTEMS

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Received 26 April 2001

We derive explicit stability conditions for delay difference equations in $\mathbb{C}^n$ (the set of n complex vectors) and estimates for the size of the solutions are derived. Applications to partial difference equations, which model diffusion and reaction processes, are given.

2000 Mathematics Subject Classification: 39A10.

1. Introduction. Stability of systems of difference equations with delays has been discussed by many authors, for example, see GiL' and Cheng [7], Zhang [11], Elaydi and Zhang [5], Pituk [10], Agarwal [1], and the references therein.

In the stability literature, we can find two major trends: stability using the first approximation Lyapunov method and the direct Lyapunov functional method. For this latter trend, see Zhang and Chen [12], Crisci et al. [4], Lakshmikantham and Trigiante [8], and Agarwal and Wong [2]. By this method many very strong results are obtained. But finding Lyapunov's functionals is usually difficult.

In this paper, we consider a class of perturbed difference equations with several delays and, by means of a Gronwall inequality and the recent estimates for the powers A^k of a constant matrix A established in [6, Theorem 1.2.1] we derive explicit stability conditions. Further, we apply our main result to an abstract partial difference equation which models reaction and diffusion processes.

2. Preliminary facts. Let $\mathbb{C}^n$ be the set of n complex vectors endowed with a norm $\|\cdot\|$. Let A be an $n\times n$-complex matrix.

Consider in $\mathbb{C}^n$ the equation

$$u_{j+1} = Au_j + f_j(u_{j-\sigma_1},\dots,u_{j-\sigma_p}),\quad j=0,1,\dots, \tag{2.1}$$

where $p \ge 1$, and $\sigma_1,\sigma_2,\dots,\sigma_p$ are nonnegative integers such that $0=\sigma_1<\sigma_2<\cdots<\sigma_p$, $\sigma_i\in\mathbb{Z}^+$, and $\mathbb{Z}^+$ is the set of nonnegative integers, f_j maps $\mathbb{C}^{np}$ into $\mathbb{C}^n$, for $j=0,1,2,\dots$.

We consider (2.1) subject to the initial conditions

$$u_j=\tau_j,\quad j=-\sigma_p,-\sigma_p+1,\dots,0. \tag{2.2}$$

It is assumed that there are nonnegative sequences q_l $(l=1,2,3,\dots,p)$ such that

$$\|f_j(u_{j-\sigma_1},\dots,u_{j-\sigma_p})\| \le \sum_{l=1}^{p} q_l(j)\|u_{j-\sigma_l}\|^m,\quad j=0,1,\dots \tag{2.3}$$

and m is a fixed positive real number.

Unlike differential equations, discrete equations with the given initial conditions always have a solution.

In order to establish our main result, we use the following discrete Gronwall type inequality.

Theorem 2.1 (see [9]). *Assume that*

$$z(k) \le C + \sum_{i=0}^{k-1} \sum_{j=1}^{p} a_j(i) z(i-\sigma_j)^m, \quad k \in \mathbb{Z}^+, \tag{2.4}$$

where $m > 0$, $0 = \sigma_1 < \sigma_2 < \cdots < \sigma_p$, $p \ge 1$, $C > 0$, $a_j(k) \ge 0$ *for* $j = 1, 2, \ldots, p$ *and* $k \in \mathbb{Z}^+$, *and* $z(k) \le C$ *for* $k = -\sigma_p, -\sigma_p + 1, \ldots, 0$.

(a) *If* $0 < m < 1$ *and* $C \le 1$, *then*

$$z(k) \le C^{m^k} \prod_{i=0}^{k-1} \left[1 + \sum_{j=1}^{p} a_j(i)\right], \quad k \in \mathbb{Z}^+. \tag{2.5}$$

(b) *If* $m = 1$, *then*

$$z(k) \le C \prod_{i=0}^{k-1} \left[1 + \sum_{j=1}^{p} a_j(i)\right], \quad k \in \mathbb{Z}^+. \tag{2.6}$$

(c) *If* $m > 1$, *then*

$$z(k) \le \frac{C}{\left\{1 - (m-1)C^{m-1} \cdot \sum_{i=0}^{k-1} \sum_{j=1}^{p} a_j(i)\right\}^{1/(m-1)}}, \quad k \in \mathbb{Z}^+, \tag{2.7}$$

provided that

$$1 - (m-1)C^{m-1} \sum_{i=0}^{k-1} \sum_{j=1}^{p} a_j(i) > 0, \quad k \in \mathbb{Z}^+. \tag{2.8}$$

Let $\lambda_1(A), \ldots, \lambda_n(A)$ be the eigenvalues of A, including their multiplicities. We will make use of the following quantity hereafter (see [6, Chapter 1]):

$$g(A) = \left\{N^2(A) - \sum_{i=1}^{n} |\lambda_i(A)|^2\right\}^{1/2}, \tag{2.9}$$

where $N(A)$ is the Frobenius (Hilbert-Schmidt) norm of A, that is, $N^2(A) = \mathrm{Trace}(AA^*)$.

There are a number of properties of $g(A)$ which are useful (see [6]). Here, we note that if A is normal, that is, $AA^* = A^*A$, then $g(A) = 0$. If $A = (a_{ij})$ is a triangular matrix such that $a_{ij} = 0$ for $1 \le j < i \le n$, then

$$g^2(A) = \sum_{1 \le i < j \le n} |a_{ij}|^2. \tag{2.10}$$

To facilitate the description of our main result, we adopt the convention that $0! = 1$, $0^0 = 1$, and that empty sums are zero. Further, the binomial coefficient C_j^i is given by

$$C_j^i = \frac{i!}{j!(i-j)!}, \quad 0 \le j \le i. \tag{2.11}$$

As normal, but we also adopt the convention that $C_j^i = 0$ when $j < 0$ or $j > i$. We define

$$\gamma_{n,i} = \begin{cases} \sqrt{\dfrac{C_i^{n-1}}{(n-1)^i}}, & i = 0,1,2,\dots,n-1, \\ 0, & \text{if } i < 0 \text{ or } i > n-1. \end{cases} \tag{2.12}$$

Note that

$$\gamma_{n,i}^2 = \frac{(n-2)(n-3)\cdots(n-i)}{(n-1)^{i-1} i!} \le \frac{1}{i!}. \tag{2.13}$$

Finally, we denote $M = \sup_{m\ge 0} \sum_{k=0}^{n-1} C_k^m \rho^{m-k}(A) g^k(A) \gamma_{n,k}$, where $\rho(A)$ is the spectral radius of A.

3. Main result.

Now, we are in a position to establish our main result pertaining to the boundedness and convergence to zero of the solutions of (2.1) subject to the initial conditions (2.2).

Theorem 3.1. *Assume that*

(i) *there are nonnegative sequences q_l $(l = 1,2,\dots,p)$ such that*

$$\|f_j(u_{j-\sigma_1},\dots,u_{j-\sigma_p})\| \le \sum_{l=1}^{p} q_l(j) \|u_{j-\sigma_l}\|^m, \quad j = 0,1,\dots \tag{3.1}$$

for $p \ge 1$ and m a fixed positive real number,

(ii) $\sum_{k=0}^{\infty} \sum_{l=1}^{p} q_l(k) < \infty$,

(iii) $v_0 = g(A) < \infty$.

Then,

(a) *if $0 < m \le 1$ and $L = M\|\tau_0\| \le 1$, with*

$$M = \sup_{m\ge 0} \sum_{k=0}^{n-1} C_k^m \rho^{m-k}(A) g^k(A) \gamma_{n,k}, \tag{3.2}$$

every solution u_j of (2.1) and (2.2), such that $\|u_j\| \le L$ for $j = -\sigma_p, -\sigma_p+1,\dots,0$, is bounded, and $\lim_{j\to\infty} \|u_j\| = 0$ whenever $\|\tau_0\| < \delta$, for $\delta > 0$ small enough;

(b) *if $m > 1$ and*

$$\|\tau_0\| \le \left\{ \frac{r}{(m-1)M^m \sum_{k=0}^{\infty} \sum_{l=1}^{p} q_l(k)} \right\}^{1/(m-1)}, \tag{3.3}$$

for an arbitrary real number $r \in (0,1)$, every solution u_j of (2.1) and (2.2), satisfying $\|u_j\| \le L$ for $j = -\sigma_p, -\sigma_p+1,\dots,0$, is bounded.

Proof. Note first that by inductive arguments, we can prove that the unique solution $\{u_j\}_{j=-\sigma_p}^{\infty}$ of (2.1), subject to given initial values: $u_0 = \tau_0$, $u_{-1},\dots,u_{-\sigma_p}$, satisfies

$$u_j = A^j \tau_0 + \sum_{k=0}^{j-1} A^{j-k-1} f_k(u_{k-\sigma_1},\dots,u_{k-\sigma_p}), \quad j = 0,1,2,\dots. \tag{3.4}$$

Hence,

$$\begin{aligned} ||u_j|| &\le ||A^j|| ||\tau_0|| + \sum_{k=0}^{j-1} ||A^{j-k-1}|| ||f_k(u_{k-\sigma_1},\dots,u_{k-\sigma_p})|| \\ &\le ||A^j|| ||\tau_0|| + \sum_{k=0}^{j-1} ||A^{j-k-1}|| \sum_{l=1}^{p} q_l(k) ||u_{k-\sigma_l}||^m. \end{aligned} \tag{3.5}$$

Denote $\Gamma = \sup_{j\ge 0} \|A^j\|$. Thus, we have

$$\begin{aligned} ||u_j|| &\le \Gamma ||\tau_0|| + \sum_{k=0}^{j-1}\sum_{l=1}^{p} \sup_{j\ge k\ge 0} ||A^{j-k-1}|| q_l(k) ||u_{k-\sigma_l}||^m \\ &\le \Gamma ||\tau_0|| + \Gamma \sum_{k=0}^{j-1}\sum_{l=1}^{p} q_l(k) ||u_{k-\sigma_l}||^m. \end{aligned} \tag{3.6}$$

We now recall from [6] that

$$||A^j|| \le \sum_{k=0}^{\min\{j,n-1\}} C_k^j \rho^{j-k}(A) g^k(A) \gamma_{n,k} \tag{3.7}$$

which implies that

$$\Gamma = \sup_{j\ge 0} ||A^j|| \le \sup_{j\ge 0} \sum_{k=0}^{n-1} C_k^j \rho^{j-k}(A) g^k(A) \gamma_{n,k} = M. \tag{3.8}$$

Thus, it follows that

$$||u_j|| \le M ||\tau_0|| + M \sum_{k=0}^{j-1}\sum_{l=1}^{p} q_l(k) ||u_{k-\sigma_l}||^m. \tag{3.9}$$

Put $v(j) = \|u_j\|$ for $j = 0,1,2,\dots$, hence

$$v(j) \le L + M \sum_{k=0}^{j-1}\sum_{l=1}^{p} q_l(k) v(k-\sigma_l)^m, \tag{3.10}$$

where $L = M\|\tau_0\|$ and $v(j) \le L$ for $j = -\sigma_p, -\sigma_p + 1, \dots, 0$.

Case 1. If $0 < m \le 1$ and $L \le 1$, then by (2.4) and Theorem 2.1(a) we have

$$v(j) \le L^{m^j} \prod_{k=0}^{j-1} \left[1 + \sum_{l=1}^{p} q_l(k)\right] \le L^{m^j} \exp\left(M \sum_{k=0}^{\infty}\sum_{l=1}^{p} q_l(k)\right). \tag{3.11}$$

Thus, establishing that the solution u_j is bounded for $j = -\sigma_p, -\sigma_p + 1, \dots, 0$, and $\lim_{j\to\infty} \|u_j\| = 0$ whenever $\|u_0\| < \delta$, for $\delta > 0$ small enough.

Case 2. If $m > 1$, then proceeding in a similar way to Case 1, we arrive at inequality (3.10).

Hence, by Theorem 2.1(b), it follows that

$$v(j) \le \frac{L}{\left\{1-(m-1)L^{m-1}\sum_{k=0}^{j-1}\sum_{l=1}^{p} q_l(k)\right\}^{1/(m-1)}} \tag{3.12}$$

provided that

$$1-(m-1)L^{m-1}\sum_{k=0}^{j-1}\sum_{l=1}^{p} q_l(k) > 0. \tag{3.13}$$

Let $r \in (0,1)$ be an arbitrary real number. We prove that condition (3.13) holds for all τ satisfying

$$\|\tau\| \le \left\{\frac{r}{(m-1)M^m\gamma}\right\}^{1/(m-1)} =: R, \tag{3.14}$$

where

$$\gamma = \sum_{k=0}^{\infty}\sum_{l=1}^{p} q_l(k) < \infty. \tag{3.15}$$

Indeed, for all such a τ_0, we have

$$\begin{aligned}
&(m-1)M^{m-1}\|\tau_0\|^{m-1}\sum_{k=0}^{j-1}\sum_{l=1}^{p} q_l(k)\\
&\quad\le (m-1)M^{m-1}\|\tau_0\|^{m-1}\sum_{k=0}^{\infty}\sum_{l=1}^{p} q_l(k) \le r.
\end{aligned} \tag{3.16}$$

Thus,

$$1-(m-1)M^m L^{m-1}\sum_{k=0}^{j-1}\sum_{l=1}^{p} q_l(k) \ge 1-r > 0. \tag{3.17}$$

Consequently, for all τ such that $\|\tau_0\| \le R$, we have

$$\begin{aligned}
\|u_j\| &\le \frac{M\|\tau_0\|}{\left\{1-(m-1)L^{m-1}\sum_{k=0}^{j-1}\sum_{l=1}^{p} Mq_l(k)\right\}^{1/(m-1)}}\\
&\le \frac{M}{(1-r)^{1/(m-1)}}\|\tau_0\|, \quad j=0,1,2,\dots.
\end{aligned} \tag{3.18}$$

Therefore, we have the boundedness of the solution u_j of (2.1), subject to the initial conditions (2.2), concluding the proof. □

4. Application. In this section, we illustrate our main result by considering an abstract partial difference equation, which models reaction and diffusion processes (see Cheng and Medina [3]).

Consider a simple three-level discrete reaction-diffusion equation of the form

$$u_{i,j+1} = au_{i-1,j} + bu_{i,j} + cu_{i+1,j} + \sum_{l=1}^{p} q_l(j)u_{i,j-\sigma_l}, \tag{4.1}$$

defined on the set

$$\Omega = \{(i,j) \mid i = 0,1,\dots,n+1,\ j = 0,1,\dots\}, \tag{4.2}$$

where q_l $(l = 1,2,\dots,p)$ are nonnegative real sequences; $u_{i,j}$ are complex sequences, $p \geq 1$; a, b, c are real numbers; and $0 = \sigma_1 < \sigma_2 < \cdots < \sigma_p$, $\sigma_i \in \mathbb{Z}^+$.

For the sake of simplicity, Dirichlet boundary conditions of the form

$$u_{0,j} = 0 = u_{n+1,j}, \quad j = 0,1,\dots \tag{4.3}$$

will be imposed.

Given an arbitrary set of initial values $u_{i,j}$, $-\sigma_p \leq j \leq 0$, $1 \leq i \leq n$, namely

$$u_{i,j} = \tau_{i,j}, \quad -\sigma_p \leq j \leq 0,\ 1 \leq i \leq n. \tag{4.4}$$

We can successively calculate $u_{1,1}, u_{2,1}, \dots, u_{n,1}$; $u_{1,2}, \dots, u_{n,2}$; $\dots$, according to (4.1) in a unique manner. Such a double sequence: $u = \{u_{i,j} \mid i = 0,1,\dots,n+1,\ j = -\sigma_p, -\sigma_p + 1, \dots\}$ is called a solution of (4.1) subject to conditions (4.3) and (4.4). An existence and uniqueness theorem for (4.1) can thus be formulated and proved in a straightforward manner.

By designating $\operatorname{col}(u_{1,j}, u_{2,j}, \dots, u_{n,j})$ as the C^n-vector u_j, we see that a solution of (4.1), (4.3), and (4.4) can also be regarded as a vector sequence $\{u_j\}_{j=-\sigma_p}^{\infty}$. Furthermore, such a sequence satisfies the delay vector recurrence relation

$$u_{j+1} = Au_j + \sum_{l=1}^{p} q_l(j)u_{j-\sigma_l}, \tag{4.5}$$

subject to the initial conditions

$$u_j = \tau_j, \quad j = -\sigma_p, -\sigma_p + 1, \dots, 0, \tag{4.6}$$

where $\tau_j = \operatorname{col}(\tau_{1,j}, \tau_{2,j}, \dots, \tau_{n,j})$, and

$$A = \begin{bmatrix} b & c & 0 & 0 & \cdots & 0 \\ a & b & c & 0 & \cdots & 0 \\ 0 & a & b & c & \cdots & 0 \\ 0 & 0 & a & b & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & a & b \end{bmatrix}, \tag{4.7}$$

In particular, (4.5) is of the form (2.1), where

$$f_j(u_{j-\sigma_1}, \dots, u_{j-\sigma_p}) = \sum_{l=1}^{p} q_l(j)u_{i,j-\sigma_l}, \quad j = 0,1,2,\dots. \tag{4.8}$$

Theorem 4.1. *Let conditions (ii) and (iii) of Theorem 3.1 hold. Further, assume that*

$$\|f_j(u_{j-\sigma_1},\ldots,u_{j-\sigma_p})\| \le \sum_{l=1}^{p} q_l(j)\|u_{j-\sigma_l}\|, \quad j=0,1,\ldots. \tag{4.9}$$

Then, if $L = M\|\tau_0\| \le 1$, *with* $M = \sup_{j\ge 0}\sum_{k=0}^{n-1} C_k^j \rho^{j-k}(A)g^k(A)\gamma_{n,k}$, *every solution* u_j *of (4.5) and (4.6), such that* $\|u_j\| \le L$, *for* $j = -\sigma_p, -\sigma_p+1,\ldots,0$, *is bounded, and* $\lim_{j\to\infty}\|u_j\| = 0$ *whenever* $\|\tau_0\| < \delta$, *for* $\delta > 0$ *small enough.*

Proof. The proof is a direct consequence of Theorem 3.1. □

Acknowledgment. This research was supported by Fondecyt under Grant No. 1000023.

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