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SEQUENCES AND SERIES INVOLVING THE SEQUENCE OF COMPOSITE NUMBERS

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Denoting by p_n and c_n the nth prime number and the nth composite number, respectively, we prove that both the sequence $(x_n)_{n\ge 1}$, defined by $x_n = \sum_{k=1}^{n}(c_{k+1}-c_k)/k - p_n/n$, and the series $\sum_{n=1}^{\infty}(p_{c_n}-c_{p_n})/np_n$ are convergent.

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1. Introduction. We use the following notation:

(i) $\pi(x)$ is the number of prime numbers less than or equal to x,
(ii) p_n is the nth prime number,
(iii) c_n is the nth composite number; $c_1 = 4, c_2 = 6, \dots$,
(iv) $\log_2 n = \log(\log n)$.

In 1967, Bojarincev [1] estimated c_n and found out that

$$c_n = n\left(1+\frac{1}{\log n}+\frac{2}{\log^2 n}+\frac{4}{\log^3 n}+\frac{19}{2}\frac{1}{\log^4 n}+\frac{181}{6}\frac{1}{\log^5 n}+o\left(\frac{1}{\log^5 n}\right)\right). \tag{1.1}$$

For $c_n^{(1)} := c_n$ and $c_n^{(k+1)} := c_{c_n^{(k)}}$, $k \ge 1$, we can prove that

$$c_n^{(k+1)} - 2c_n^{(k)} + c_n^{(k-1)} \sim \frac{n}{\log^2 n}. \tag{1.2}$$

If n is large enough, then

$$c_n^{(k)} > \sqrt{c_n^{(k-1)} \cdot c_n^{(k+1)}}. \tag{1.3}$$

It was shown in [3] that, for n large enough we have

$$p_{c_n} > c_{p_n}. \tag{1.4}$$

Of course, the irregularities in the distribution of the prime numbers imply irregularities in the distribution of the composed numbers. Although in the sequence of the composite numbers, large "gaps" cannot be found. For the sequence $(p_n)_{n\ge 1}$ we have $\limsup_{n\to\infty}(p_{n+1}-p_n) = \infty$, while in the case of the sequence $(c_n)_{n\ge 1}$ we have $1 \le c_{k+1}-c_k \le 2$, with the specification that $c_{k+1}-c_k = 2$ if and only if the number c_{k+1} is prime. In this case, denoting $c_k+1 = p_m$, it is proved in [3] that

$$k = k(m) = p_m + mx_m \quad \text{with } \lim_{m\to\infty} x_m = 1. \tag{1.5}$$

Since $c_n \sim n$, we can expect that "in mean" the sequence $(c_n)_{n\geq 1}$ behaves as the sequence of the natural numbers. It is readily seen that, since the series $\sum_{n=1}^{\infty} 1/n(p_n - p_{n+1})$ is divergent, it follows that the series $\sum_{n=1}^{\infty} 1/c_n(p_n - p_{n+1})$ is divergent too.

The situations to be analyzed in the present paper are, however, more complicated and lean on a series of facts, namely

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) = \gamma; \tag{1.6}$$

$$\text{the series } \sum_{n=2}^{\infty}\frac{1}{n(\log n)^{\alpha}} \text{ is convergent,} \quad \text{for } \alpha > 1; \tag{1.7}$$

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O\left(\frac{x}{\log^k x}\right), \quad \text{for every } k > 0; \tag{1.8}$$

$$p_n = n\left(\log n + \log_2 n - 1 + \frac{\log_2 n - 2}{\log n} + O\left(\frac{\log_2^2 n}{\log^2 n}\right)\right) \quad \text{(see [2]);} \tag{1.9}$$

$$\log p_n = \log n + \log_2 n + \frac{\log_2 n - 1}{\log n} + O\left(\frac{\log_2^2 n}{\log^2 n}\right) \quad \text{(see [5]);} \tag{1.10}$$

$$b = \lim_{n\to\infty}\left(\sum_{k=2}^{n}\frac{1}{p_k} - \log\log n\right) \quad \text{(see [4]).} \tag{1.11}$$

2. Asymptotic behavior of certain series

Proposition 2.1. *The sequence $x_n = \sum_{k=1}^{n}(c_{k+1} - c_k)/k - p_n/n$ is convergent.*

Proof. We have

$$x_n = \sum_{k=1}^{n}\frac{1}{k} + \sum_{k\leq n}{}'\frac{1}{k} - \frac{p_n}{n}, \tag{2.1}$$

where $\sum'$ extends to all values of k such that $c_k + 1$ is a prime number, that is, $c_k + 1 = p_m$, $m = 3, 4, \ldots, \pi(n)$. It follows by (1.5) that

$$\sum_{k=1}^{n}{}'\frac{1}{k} = \sum_{m=3}^{\pi(n)}\frac{1}{p_m + m x_m} = \sum_{m=3}^{\pi(n)}\frac{1}{p_m} - \sum_{m=3}^{\pi(n)}\frac{m x_m}{p_m(p_m + m x_m)}. \tag{2.2}$$

Since $p_m \sim m\log m$, we get $m x_m / p_m(p_m + m x_m) \sim 1 m \log^2 m$ and then (1.7) implies that the series $\sum_{m=1}^{\infty} m x_m / p_m(p_m + m x_m)$ is convergent; denote its sum by a. In view of (1.11), it then follows that

$$\sum_{k=1}^{n}{}'\frac{1}{k} - \log_2 n \xrightarrow{n\to\infty} b - a. \tag{2.3}$$

By (1.10) $\lim_{n\to\infty}(p_n/n - \log n - \log_2 n) = -1$, hence (1.6) and (2.3) imply that

$$\lim_{n\to\infty} x_n = \gamma + b - a + 1. \tag{2.4}$$

The proof is completed. □

Remark 2.2. It follows by Proposition 2.1 that both the series $\sum_{k=1}^{\infty}(c_{k+1}-c_k)/k$ and $\sum_{k=1}^{\infty}(c_{k+1}-c_k-1)/k$ are divergent.

Proposition 2.3. *Let s be a real number. If $y_n = \sum_{k=1}^{n}(c_{k+1}-c_k)^s/k$, then $y_n = \log n + (2^s-1)\log_2 n + O(1)$.*

Proof. If the number c_k+1 is composed, then $c_{k+1}-c_k = 1$, while if c_k+1 is prime then $c_{k+1}-c_k = 2$. Thus

$$y_n = \sum_{k=1}^{n}\frac{1}{k} + \sum_{k=1}^{n}{}' \frac{2^s-1}{k}, \tag{2.5}$$

where $\sum'$ extends to the indices k such that c_k+1 is prime. Now (1.6) and (2.3) imply that

$$y_n = \log n + \gamma + o(1) + (2^s-1)(\log_2 n + b - a + o(1)) \tag{2.6}$$

and the proof ends. □

Proposition 2.4. *If $z_n = \sum_{k=1}^{n}(c_{k+1}-c_k)/(c_{k+2}-c_{k+1})$, then $z_n = n + 3/2\cdot n/\log n + O(n/\log^2 n)$.*

Proof. The following cases can arise:

(a) if both c_{k+1} and $c_{k+1}+1$ are composite numbers, then $c_{k+1}-c_k = c_{k+2}-c_{k+1} = 1$;
(b) if c_{k+1} is prime and $c_{k+1}+1$ is composed, then $c_{k+1}-c_k = 2$ and $c_{k+2}-c_{k+1} = 1$;
(c) if c_{k+1} is composite and $c_{k+1}+1$ is prime, then $c_{k+1}-c_k = 1$ and $c_{k+2}-c_{k+1} = 2$;
(d) if both c_{k+1} and $c_{k+1}+1$ are prime numbers, then $c_{k+1}-c_k = c_{k+2}-c_{k+1} = 2$.

Next, denote by $\pi_2(x)$ the number of the prime numbers $p \le x$ such that $p+2$ is prime. It is known (see [2]) that

$$\pi_2(x) = O\left(\frac{x}{\log^2 x}\right). \tag{2.7}$$

By taking into consideration the above four cases, it follows that

$$z_n = 1(n-\pi(n)) + 2(\pi(n)-\pi_2(n)) + \frac{1}{2}(\pi(n)-\pi_2(n)) + \pi_2(n) + O(1). \tag{2.8}$$

Since $\pi(x) = x/\log x + O(x/\log^2 x)$, the desired conclusion follows. □

Proposition 2.5. *If $t_n = \sum_{k=1}^{n}(c_{k+2}-2c_{k+1}+c_k)^2$, then $t_n = 2\pi(n) + O(1)$.*

Proof. By analyzing the cases occurring in the proof of the preceding proposition, we see that $c_{k+2}-2c_{k+1}+c_k = 0$ in the cases (a) and (d), while $(c_{k+2}-2c_{k+1}+c_k)^2 = 1$ in the cases (b) and (d). Consequently, $t_n = 2\pi(n) + O(1)$. □

3. Studying the convergence of certain series

Proposition 3.1. *The series*

$$\sum_{n=1}^{\infty}\frac{p_{c_n}-c_{p_n}}{np_n} \tag{3.1}$$

is convergent.

First we prove the following fact.

Lemma 3.2. *The relation*

$$\frac{1}{\log p_n} = \frac{1}{\log n} - \frac{\log_2 n}{\log^2 n} + O\left(\frac{\log_2^2 n}{\log^3 n}\right) \tag{3.2}$$

holds.

Proof. By (1.10), it follows that

$$\frac{1}{\log p_n} = \frac{1}{\log n} \cdot \frac{1}{1+\log_2 n/\log n + \log_2 n/\log^2 n + O(1/\log^2 n)}. \tag{3.3}$$

For $|x| < 1$ we have $1/(1+x) = 1-x+O(x^2)$, hence,

$$\frac{1}{\log p_n} = \frac{1}{\log n} \cdot \left(1 - \frac{\log_2 n}{\log n} - \frac{\log_2 n}{\log^2 n} + O\left(\frac{\log_2^2 n}{\log^2 n}\right)\right) \tag{3.4}$$

which implies the desired conclusion. □

Proof of Proposition 3.1. By (1.1), we get

$$c_{p_n} = p_n\left(1 + \frac{1}{\log p_n} + \frac{2}{\log^2 p_n} + O\left(\frac{1}{\log^3 p_n}\right)\right) \tag{3.5}$$

and then we have, in view of the above lemma,

$$c_{p_n} = p_n\left(1 + \frac{1}{\log n} - \frac{\log_2 n - 2}{\log^2 n} + O\left(\frac{\log_2^2 n}{\log^3 n}\right)\right). \tag{3.6}$$

Now, replacing p_n by means of (1.9), we get

$$c_{p_n} = p_n + n + \frac{n}{\log n} + O\left(\frac{n\log_2^2 n}{\log^2 n}\right). \tag{3.7}$$

If we let $x = p_{c_n}$ in (1.8), then we deduce

$$c_n = \int_2^{p_{c_n}} \frac{dt}{\log t} + O\left(\frac{n}{\log^2 n}\right). \tag{3.8}$$

For $x = c_{p_n}$, also (1.8) implies

$$\pi(c_{p_n}) = \int_2^{c_{p_n}} \frac{dt}{\log t} + O\left(\frac{n}{\log^2 n}\right). \tag{3.9}$$

It is readily seen that $\pi(c_m)+m+1 = c_m$ and for $m = p_n$ this implies that $\pi(c_{p_n}) + p_n + 1 = c_{p_n}$. Then, (3.9) becomes

$$c_{p_n} - p_n - 1 = \int_2^{c_{p_n}} \frac{dt}{\log t} + O\left(\frac{n}{\log^2 n}\right). \tag{3.10}$$

By subtracting (3.8) from the above relation, we get

$$c_{p_n} - p_n - c_n - 1 = \int_{p_{c_n}}^{c_{p_n}} \frac{dt}{\log t} + O\left(\frac{n}{\log^2 n}\right). \tag{3.11}$$

In view of (1.1) and (3.7), this relation takes the form

$$O\left(\frac{n\log_2^2 n}{\log^2 n}\right)=\int_{p_{c_n}}^{c_{p_n}}\frac{dt}{\log t}+O\left(\frac{n}{\log^2 n}\right). \tag{3.12}$$

By applying the mean value theorem to the above integral, it follows that there exists $\theta_n \in (p_{c_n}, c_{p_n})$ such that

$$\int_{p_{c_n}}^{c_{p_n}}\frac{dt}{\log t}=\frac{c_{p_n}-p_{c_n}}{\log\theta_n}. \tag{3.13}$$

Since $\log\theta_n \sim \log n$, (3.12) becomes

$$O\left(\frac{n\log_2^2 n}{\log^2 n}\right)=\frac{c_{p_n}-p_{c_n}}{\log n}+O\left(\frac{n}{\log^2 n}\right). \tag{3.14}$$

Since $p_n \sim n\log n$, we obtain furthermore

$$\frac{c_{p_n}-p_{c_n}}{np_n}=O\left(\frac{\log_2^2 n}{n\log^2 n}\right). \tag{3.15}$$

For n large enough we have

$$c_{p_n}-p_{c_n}>0, \qquad \frac{\log_2^2 n}{n\log^2 n}<\frac{1}{n\log^{1.5} n}, \tag{3.16}$$

hence in view of (1.6), it follows that the series $\sum_{n=1}^{\infty}(c_{p_n}-p_{c_n})/np_n$ is convergent. □

Proposition 3.3. *Let $k\ge 3$ and let $\alpha_1,\alpha_2,\dots,\alpha_k$ be real numbers such that $\sum_{i=1}^{k}\alpha_i=0$ and $\sum_{i=1}^{k}(i-1)\alpha_i=0$. If $(x_n)_{n\ge 1}$ is a decreasing sequence which converges to 0, then the series*

$$\sum_{n=1}^{\infty}\left(\alpha_k c_{n+k}+\alpha_{k-1}c_{n+k-1}+\cdots+\alpha_1 c_{n+1}\right)x_n \tag{3.17}$$

is convergent.

Proof. Denote $\varepsilon_i=\alpha_k c_{i+k}+\alpha_{k-1}c_{i+k-1}+\cdots+\alpha_1 c_{i+1}$ and $a=\alpha_1 c_2+(\alpha_1+\alpha_2)c_3+(\alpha_1+\alpha_2+\cdots+\alpha_{k-1})c_k$. Since $\sum_{i=1}^{k}\alpha_i=0$, it then follows that

$$\sum_{i=1}^{k}\varepsilon_i=a+\alpha_k c_{n+k}+(\alpha_k+\alpha_{k-1})c_{n+k-1}+\cdots+(\alpha_k+\alpha_{k-1}+\cdots+\alpha_2)c_{n+2}. \tag{3.18}$$

Denote $c_{n+i}=c_{n+2}+x_i^{(n)}$. Since $1\le c_{i+1}-c_i\le 2$, it then follows that $1\le x_i^{(n)}<2k$. Relation (3.18) can take the form

$$\begin{aligned}\sum_{i=1}^{k}\varepsilon_i&=a+\sum_{i=1}^{k}(i-1)\alpha_i c_{n+2}+\sum_{i=3}^{k}\left(\alpha_k+\alpha_{k-1}+\cdots+\alpha_i\right)x_i^{(n)}\\&=a+\sum_{i=3}^{k}\left(\alpha_k+\alpha_{k-1}+\cdots+\alpha_i\right)x_i^{(n)}.\end{aligned} \tag{3.19}$$

Since

$$\left|\sum_{i=3}^{k}(\alpha_k+\alpha_{k-1}+\cdots+\alpha_i)x_i^{(n)}\right| \le \sum_{i=3}^{k}(|\alpha_k|+|\alpha_{k-1}|+\cdots+|\alpha_i|)\cdot 2k = M, \quad (3.20)$$

then it follows that the sequence $(\sum_{i=1}^{n}\varepsilon_i)_{n\ge 1}$ is bounded. Consequently, the convergence of our series follows by Dirichlet's criterion. □

Proposition 3.4. *The series*

$$\sum_{n=1}^{\infty}\frac{c_{n+1}-c_n-1}{p_n} \quad (3.21)$$

is convergent.

Proof. Denoting $S_n=\sum_{k=1}^{n}(c_{k+1}-c_k-1)/p_k$, it follows that $S_n=\sum'_{p_k\le n}1/p_k$, where $\sum'$ extends to the indices k such that $c_k+1=p_m$ with prime p_m. In view of (1.5), it follows that $k=k(m)\sim p_m\sim m\log m$, hence $p_k\sim k\log k\sim m\log^2 m$. then (1.7) implies that the series $\sum_{n=1}^{\infty}(c_{n+1}-c_n-1)/p_n$ is convergent. □

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