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THE MONAD INDUCED BY THE HOM-FUNCTOR IN THE CATEGORY OF TOPOLOGICAL SPACES AND ITS ASSOCIATED EILENBERG-MOORE ALGEBRAS

KOENA R. NAILANA

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We discuss the monad associated with the topology of pointwise convergence. We also study examples of the Eilenberg-Moore algebras for this monad.

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1. Introduction.

Let Top denote the category of topological spaces and continuous functions. Let $\mathbb{R}$ denote the real line with the usual topology, and for each topological space X, let $C(X,\mathbb{R})$ be the set of continuous real-valued functions from X to $\mathbb{R}$. Consider the contravariant hom-functor $C_p : \text{Top} \to \text{Top}^{\text{op}}$ defined by assigning to each space X the space of continuous real-valued functions with the topology of pointwise convergence. We denote this space by C_pX. The space C_pX has been extensively studied. A fundamental reference on C_pX is Arkhangel'skiĭ [2]. We recall that the subbasic open sets of C_pX are sets of the form $[f,V]$, where $[x,V] = \{f \in C_pX : f(x) \in V,\ V \text{ open in } \mathbb{R}\}$.

2. The monad induced by the hom-functor in Top and the associated M-algebras.

We now consider the composite functor $C_p^{\text{op}}C_p : \text{Top} \to \text{Top}^{\text{op}} \to \text{Top}$ where C_p^{op} is the dual functor. Let $M = C_p^{\text{op}}C_p$. If $x \in X$, then the function $\hat{x} : C_pX \to \mathbb{R}$ defined by $\hat{x}(f) = f(x)$ is called the evaluation map at x. The following propositions are important since they ensure that our morphisms are continuous. The proofs are straightforward and will be omitted.

Proposition 2.1. (i) *For all $x \in X$, $\hat{x} : C_pX \to \mathbb{R}$ is continuous.*
(ii) *For all $g \in C_pX$, $\hat{g} : MC_pX \to \mathbb{R}$ is continuous.*

Proposition 2.2. *Let X be any topological space. Then*
(i) *$\eta_X : X \to MX$, where $\eta_X(x) = \hat{x}$ is continuous.*
(ii) *$\mu_X : MMX \to MX$, where $\mu_X(\gamma)[g] = \gamma(\hat{g})$ is continuous.*

We recall from [1] that a monad on a category $\mathbb{A}$ is a triplet $\mathbb{M} = (M,\eta,\mu)$ consisting of a functor $M : \mathbb{A} \to \mathbb{A}$ and natural transformations $\eta : \text{id}_{\mathbb{A}} \to M$ and $\mu : MM \to M$ such that $\mu \circ M\mu = \mu \circ \mu M$, $\mu \circ M\eta = \text{id}$, and $\mu \circ \eta M = \text{id}$.

Proposition 2.3. *The triplet (M,η,μ), where $\eta : \text{id}_{\text{Top}} \to M$ and $\mu : MM \to M$ are defined by $\eta_X(x) = \hat{x}$ and $\mu_X(\gamma)[g] = \gamma(\hat{g})$, respectively, where $x \in X$, $g \in C_pX$, is a monad.*

Proof. We first check that $\eta : \mathrm{id}_{\mathrm{Top}} \to M$ and $\mu : MM \to M$ are natural transformations. Let $f : X \to Y$ be a continuous function. We show that $M(f) \circ \eta_X = \eta_Y \circ f$. We define η_X and $M(f)$ by $\eta_X(x) = \hat{x}$ and $M(f)(\gamma)[g] = \gamma(g \circ f)$ where $g : Y \to \mathbb{R}$ is continuous, $\gamma \in M(X)$, and $\hat{}$ denotes evaluation, for example, $\hat{x}(g) = g(x)$. Then $M(f) \circ \eta_X(x) = M(f)(\hat{x})$. Let $g \in C_pX$. Then $M(f)(\hat{x})[g] = \hat{x}(g \circ f) = g \circ f(x) = g(f(x)) = \widehat{f(x)}[g]$. Hence $M(f)(\hat{x}) = \widehat{f(x)}$. Now $\eta_Y \circ f(x) = \eta_Y(f(x)) = \widehat{f(x)}$. Let $g \in C_pX$. Then $\widehat{f(x)}[g] = g(f(x))$. Hence $M(f) \circ \eta_X = \eta_Y \circ f$. We define μ_X by $\mu_X(\gamma)[g] = \gamma(\hat{g})$ where $\gamma \in MM(X)$, $g \in C_pX$, and $\hat{g}$ denotes the evaluation function at g, that is, $\hat{g} : M(X) \to \mathbb{R}$. We now show that $\mu : MM \to M$ is a natural transformation, that is, $M(f) \circ \mu_X = \mu_Y \circ M^2(f)$. Let $h \in C_pY$. Then

$$(M(f) \circ \mu_X)(\gamma)[h] = M(f)(\mu_X(\gamma))[h] = \mu_X(\gamma)(h \circ f) = \gamma(\widehat{h \circ f}). \tag{2.1}$$

On the other hand,

$$\begin{aligned} \mu_Y \circ M^2(f)(\gamma)[h] &= \mu_Y(M^2(f)(\gamma))[h] \\ &= M^2(f)(\gamma)(\hat{h}) \\ &= M(M(f))(\gamma)(\hat{h}) \\ &= \gamma(\hat{h} \circ M(f)). \end{aligned} \tag{2.2}$$

Let $\lambda : C_pX \to \mathbb{R}$. Then $(\hat{h} \circ M(f))(\lambda) = \hat{h}(M(f)(\lambda)) = M(f)(\lambda)[h] = \lambda(h \circ f) = \widehat{h \circ f}(\lambda)$. Therefore, $\hat{h} \circ M(f) = \widehat{h \circ f}$. From the equations

$$\begin{gathered} M(f) \circ \mu_X(\gamma)[h] = \gamma(\widehat{h \circ f}), \\ \mu_Y \circ M^2(f)(\gamma)[h] = \gamma(\hat{h} \circ M(f)), \\ (\hat{h} \circ M(f))(\lambda) = \widehat{h \circ f}(\lambda), \end{gathered} \tag{2.3}$$

we get $M(f) \circ \mu_X = \mu_Y \circ M^2(f)$. Therefore $\mu : MM \to M$ is a natural transformation. We now show that the other monad conditions are satisfied. First, we show that $\mu_X \circ M\eta = \mathrm{id}$. We prove that $\mu_X \circ M\eta = \mathrm{id}$ and $\mu_X \circ \eta_M = \mathrm{id}$. Let $\gamma \in M(X)$ and $f \in C_pX$. Then $\hat{f} : M(X) \to \mathbb{R}$ and $(\mu_X \circ M\eta)(\gamma) \in M(X)$. Then $(\mu_X \circ M\eta)(\gamma)[f] = \mu_X(M\eta(\gamma))[f] = M\eta(\gamma)(\hat{f}) = \gamma(\hat{f} \circ \eta) = \gamma[f]$. Therefore $\mu_X \circ M\eta = \mathrm{id}$. On the other hand $(\mu_X \circ \eta_M)(\gamma)[f] = \mu_X(\eta_M)(\gamma)[f] = \eta_M(\gamma)[\hat{f}] = \hat{\gamma}(\hat{f}) = \hat{f}(\gamma) = \gamma(f)$. Therefore $\mu_X \circ \eta_M = \mathrm{id}$. Second, we show that $\mu_X \circ \mu_M = \mu_X \circ M\mu_X$. We prove that $\mu_X \circ \mu_M = \mu_X \circ M\mu_X$. Let $\gamma \in MMM(X)$. Then $\mu_X \circ M\mu_X(\gamma) \in M(X)$. Let $f \in C_pX$. Then $(\mu_X \circ M\mu_X)(\gamma)[f] = \mu_X(M\mu_X(\gamma))[f] = M\mu_X(\gamma)[\hat{f}] = \gamma(\hat{f} \circ \mu_X) = \gamma(\hat{\hat{f}})$. On the other hand, $(\mu_X \circ \mu_M)(\gamma)[f] = \mu_X(\mu_M(\gamma))[f] = \mu_M(\gamma)(\hat{f}) = \gamma(\hat{\hat{f}})$. Therefore $\mu_X \circ \mu_M = \mu_X \circ M\mu_X$. Therefore (M, η, μ) is a monad. □

If $\mathbb{M} = (M, \eta, \mu)$ is a monad on $\mathbb{A}$, then (A, h_A) is called an Eilenberg-Moore algebra or simply an M-algebra if the algebra map $h_A : MA \to A$ satisfies $h_A \circ \eta_A = \mathrm{id}_A$ and $h_A \circ Mh_A = h_A \circ \mu_A$.

We now look at examples of the M-algebras of the monad (M, η, μ).

Proposition 2.4. *The real line $\mathbb{R}$ is an M-algebra.*

Proof. We define $h_{\mathbb{R}} : M\mathbb{R} \to \mathbb{R}$ as $\hat{\mathbf{1}}_{\mathbb{R}}$, that is, the identity map with respect to $\mathbb{R}$, and show that the M-algebra conditions are satisfied. It is obvious that the map $h_{\mathbb{R}}$ is continuous. Let $x \in \mathbb{R}$. Then $h_{\mathbb{R}} \circ \eta_{\mathbb{R}}(x) = h_{\mathbb{R}}(\hat{x}) = \hat{x}(\mathbf{1}_{\mathbb{R}}) = \mathbf{1}_{\mathbb{R}}(x)$. Therefore $h_{\mathbb{R}} \circ \eta_{\mathbb{R}} = \mathbf{1}_{\mathbb{R}}$. Now let $\gamma \in MM(\mathbb{R})$. Then $h_{\mathbb{R}} \circ \mu_{\mathbb{R}}(\gamma) = \hat{\mathbf{1}}_{\mathbb{R}}(\mu_{\mathbb{R}}(\gamma)) = \mu_{\mathbb{R}}(\gamma)(\mathbf{1}_{\mathbb{R}}) = \gamma(\hat{\mathbf{1}}_{\mathbb{R}})$. On the other hand, $h_{\mathbb{R}} \circ Mh_{\mathbb{R}}(\gamma) = \hat{\mathbf{1}}_{\mathbb{R}}(M\hat{\mathbf{1}}_{\mathbb{R}}(\gamma)) = \hat{\mathbf{1}}_{\mathbb{R}}(\gamma \circ C_p(\hat{\mathbf{1}}_{\mathbb{R}})) = \gamma \circ C_p(\hat{\mathbf{1}}_{\mathbb{R}})(\mathbf{1}_{\mathbb{R}}) = \gamma(C_p(\hat{\mathbf{1}}_{\mathbb{R}})(\mathbf{1}_{\mathbb{R}})) = \gamma(\mathbf{1}_{\mathbb{R}} \circ \hat{\mathbf{1}}_{\mathbb{R}}) = \gamma(\hat{\mathbf{1}}_{\mathbb{R}})$. Therefore $h_{\mathbb{R}} \circ \mu_{\mathbb{R}} = h_{\mathbb{R}} \circ Mh_{\mathbb{R}}$. □

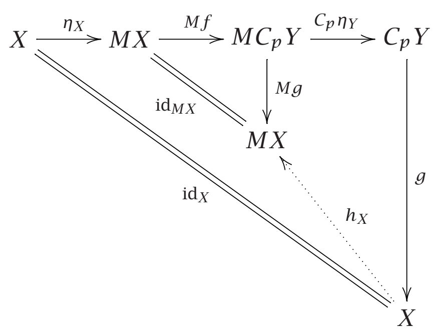
Proposition 2.5. *For each $X \in \mathrm{Top}$, C_pX is an M-algebra with $h_{C_pX} = C_p(\eta_X)$.*

Proof. We first define $h_{C_pX} : MC_pX \to C_pX$. Let $\varphi \in MC_pX$. We define h_{C_pX} by $h_{C_pX}(\varphi) = \varphi \circ \eta_X = C_p\eta_X(\varphi)$. Then the map h_{C_pX} is continuous, since it is the composite of continuous functions φ and η_X. We now show that the conditions for an M-algebra are satisfied. Thus, we must show that $h_{C_pX} \circ \eta_{C_pX} = \mathrm{id}_{C_pX}$. Let $f \in C_pX$. Then $h_{C_pX} \circ \eta_{C_pX}(f) = h_{C_pX}(\eta_{C_pX}(f)) = C_p\eta_X(\hat{f}) = \hat{f} \circ \eta_X = f = \mathrm{id}_{C_pX}(f)$, since $\hat{f} \circ \eta_X(x) = \hat{f}(\eta_X(x)) = \hat{x}(f) = f(x)$. Therefore $h_{C_pX} \circ \eta_{C_pX} = \mathrm{id}_{C_pX}$.

We must now show that $h_{C_pX} \circ \mu_{C_pX} = h_{C_pX} \circ Mh_{C_pX}$. Let $\gamma \in MMC_pX$. Then $h_{C_pX} \circ \mu_{C_pX}(\gamma) = h_{C_pX}(\mu_{C_pX}(\gamma)) = C_p\eta_X(\mu_{C_pX}(\gamma)) = \mu_{C_pX}(\gamma) \circ \eta_X$. Now let $x \in X$. Then $\mu_{C_pX}(\gamma) \circ \eta_X(x) = \mu_{C_pX}(\gamma)(\hat{x}) = \gamma(\hat{\hat{x}})$. On the other hand, $h_{C_pX} \circ Mh_{C_pX}(\gamma) = C_p\eta_X \circ MC_p\eta_X(\gamma) = C_p(M\eta_X \circ \eta_X)(\gamma) = \gamma \circ M\eta_X \circ \eta_X$. Let $x \in X$. Then $M\eta_X \circ \eta_X(x) = M\eta_X(\hat{x}) = \hat{x} \circ C_p\eta_X = \hat{\hat{x}}$. Therefore $h_{C_pX} \circ \mu_{C_pX} = h_{C_pX} \circ Mh_{C_pX}$. Hence C_pX is an M-algebra. □

Proposition 2.6. *Retracts of C_pX are M-algebras.*

Proof. Let $g : C_pY \to X$ be a retraction. Then there is a continuous function $f : X \to C_pY$ such that $g \circ f = \mathrm{id}_X$. The following diagram will help us define the algebra map $h_X : MX \to X$:



(2.4)

Define

$$h_X = g \circ C_p\eta \circ Mf = g \circ C_p(C_pf \circ \eta_Y). \tag{2.5}$$

Since h_X is the composite of continuous functions, then it is continuous.

Now,

$$\begin{aligned} h_X \circ \eta_X(x) &= h_X(\hat{x}) = g(C(Cf \circ \eta_Y)(\hat{x})) = g(\hat{x} \circ Cf \circ \eta_Y) \\ &= g(\widehat{f(x)} \circ \eta_Y) = g(f(x)) = \mathrm{id}_X(x), \end{aligned} \tag{2.6}$$

since g is a retraction.

We now show that $h_X \circ \mu_X = h_X \circ Mh_X$. Let $y \in MMX$. Then

$$\begin{aligned} h_X \circ \mu_X(y) &= h_X(\mu_X(y)) = g \circ C_p(C_p f \circ \eta_Y)(\mu_X(y)) \\ &= g(C_p(C_p f \circ \eta_Y)(\mu_X(y))) = g(\mu_X(y) \circ C_p f \circ \eta_Y). \end{aligned} \tag{2.7}$$

If $k \in C_p X$, then

$$\mu_X(y)(k) = y(\hat{k}). \tag{2.8}$$

On the other hand,

$$\begin{aligned} h_X \circ Mh_X &= g \circ C_p(C_p f \circ \eta_Y) \circ M(g \circ C_p(C_p f \circ \eta_Y)) \\ &= g \circ C_p \eta_Y \circ Mf \circ Mg \circ MC_p(C_p f \circ \eta_Y) \\ &= g \circ C_p \eta_Y \circ M(f \circ g) \circ MC_p(C_p f \circ \eta_Y) \\ &= g \circ C_p \eta_Y \circ M(\mathrm{id}_X) \circ MC_p(C_p f \circ \eta_Y) \\ &= g \circ C_p \eta_Y \circ MC_p(C_p f \circ \eta_Y) \\ &= g \circ C_p(M(C_p f \circ \eta_Y) \circ \eta_Y) \\ &= g \circ C_p(\eta_{C_p X} \circ C_p f \circ \eta_Y). \end{aligned} \tag{2.9}$$

Now,

$$\begin{aligned} h_X \circ Mh_X(y) &= g(C_p(\eta_{C_p X} \circ C_p f \circ \eta_Y)(y)) \\ &= g(y \circ \eta_{C_p X} \circ C_p f \circ \eta_Y). \end{aligned} \tag{2.10}$$

We only need to show that $y \circ \eta_{C_p X} = \mu_X$. Let $k \in C_p X$. Then $y \circ \eta_{C_p X}(k) = y(\eta_{C_p X}(k)) = y(\hat{k})$. From (2.8), we have $y \circ \eta_{C_p X} = \mu_X$ and therefore $h_X \circ \mu_X = h_X \circ Mh_X$. Hence retracts of $C_p X$ are M-algebras. □

3. The algebra morphisms and the transfer of ring structure from MX to X for an M-algebra (X, h_X).

For an M-algebra (X, h_X) the ring structure on MX can be transferred to X, via h_X, in such a way that X becomes a ring with respect to the induced operations.

Definition 3.1. On an M-algebra (X, h_X) define
(i) $x_1 + x_2$ to be $h_X(\eta_X(x_1) + \eta_X(x_2))$,
(ii) $x_1 \cdot x_2$ to be $h_X(\eta_X(x_1) \cdot \eta_X(x_2))$.

In addition to the ring structure defined above we also define the scalar multiplication in the following way: define tx to be $h_X(t\eta_X(x))$, where t is a scalar.

According to Definition 3.1, $C_p X$ (being an M-algebra, Proposition 2.5) has now two concepts of the operations "+" and "·", the natural one defined pointwise

$$(h_X(x+y) = h_X(x) + h_X(y), h_X(xy) = h_x(x) h_X(y)) \tag{3.1}$$

and Definition 3.1. The same applies to MX. We omit the straightforward proof of the following proposition.

Proposition 3.2. *The natural operations on MX defined pointwise coincide with the corresponding ones defined above.*

Lemma 3.3. *The topology on X is initial with respect to η_X, that is, X has the weak topology induced by η_X into $C_pC_pX = MX$.*

Proof. Basic neighborhoods of $\eta_X(x)$ have inverse images of η_X of the form $\cap_{i=1}^n f_i^{-1}[W_i]$. □

Lemma 3.4 [2]. *Let $\varphi \in C_pC_pX$ such that $\varphi : (C_pX, h_{C_pX}) \to (\mathbb{R}, h_{\mathbb{R}})$ is a linear functional. Then there are $x_1,\dots,x_n \in X$, $\lambda_1,\dots,\lambda_n \in \mathbb{R}$ such that $\varphi = \sum_{i=1}^n \lambda_i \hat{x}_i$.*

Proposition 3.5. *If $\varphi : (C_pX, h_{C_pX}) \to (\mathbb{R}, h_{\mathbb{R}})$ is a nontrivial continuous multiplicative linear functional, then there is $x \in X$ such that $\varphi = \hat{x}$, that is, φ is a point evaluation.*

Proof. By Lemma 3.4, there are points $x_1,\dots,x_n \in X$, and scalars $\lambda_1,\dots,\lambda_n \in \mathbb{R}$ such that $\varphi = \sum_{i=1}^n \lambda_i \hat{x}_i$ where $\lambda_i = \varphi(g_i)$, $g_i \in C_pX$ being such that $g_i(x_i) = 1$, $g_i(x_j) = 0$ for $i \neq j$, $0 \le g_i \le 1$. Now $\varphi(g_k^2) = \varphi(g_k)^2 = \lambda_k$. Also $\varphi(g_k^2) = \sum_{i=1}^n \lambda_i \hat{x}_i(g_k^2) = \sum_{i=1}^n \lambda_i g_k^2(x_i) = \lambda_k$.

Thus $\lambda_k = \lambda_k^2$, so that $\lambda_k = 0$ or $\lambda_k = 1$ for $k = 1,2,\dots,n$. Moreover, $\lambda_k = g_k(x_k) \ge 0$. Furthermore, $\varphi(\mathbf{1}) = 1$ gives $1 = \varphi(\mathbf{1}) = \sum_{i=1}^n \lambda_i \hat{x}_i(\mathbf{1}) = \sum_{i=1}^n \lambda_i$. Consequently, all λ_i's except one are zero, the exceptional one being one 1. Let $x = x_l$, where $\lambda_l = 1$. Then $\lambda_i = 0$ for $i \neq l$, so that $\varphi = \lambda_l \hat{x}_l = \hat{x}_l$. □

Proposition 3.6. *Let $\varphi : (C_pX, h_{C_pX}) \to (\mathbb{R}, h_{\mathbb{R}})$ be an algebra map. Then φ is a continuous ring homomorphism.*

Proof. Given $f, g \in C_pX$, consider $\eta_{C_pX}(f) + \eta_{C_pX}(g)$ in MC_pX. We have

$$h_{\mathbb{R}} \circ C^2\varphi(\eta_{C_pX}(f) + \eta_{C_pX}(g)) = h_{\mathbb{R}} \circ M\varphi(\eta_{C_pX}(f)) + h_{\mathbb{R}} \circ M\varphi(\eta_{C_pX}(g)) \tag{3.2}$$

by Lemma 3.4.

Hence $\varphi \circ h_X(\eta_{C_pX}(f) + \eta_{C_pX}(g)) = \varphi \circ h_X(\eta_{C_pX}(f)) + \varphi \circ h_X(\eta_{C_pX}(g))$, so that $\varphi(f+g) = \varphi(f) + \varphi(g)$, since h_X preserves the ring structure. Similarly, $\varphi(f \cdot g) = \varphi(f) \cdot \varphi(g)$. We also have $\varphi(tf) = t\varphi(f)$, $t \in \mathbb{R}$. Moreover $\varphi(\mathbf{1}) = 1$, where $\mathbf{1}$ denotes the constant function with value equal to 1. □

Proposition 3.7. *Every algebra map $\varphi : (C_pX, h_{C_pX}) \to (\mathbb{R}, h_{\mathbb{R}})$ is a point evaluation map.*

Proof. By the above proposition, φ is a continuous ring homomorphism, that is, a continuous multiplicative linear functional. Thus, there is some $x \in X$ such that $\varphi(f) = f(x)$ for all f in C_pX, by the above proposition. □

Theorem 3.8. *The algebra morphisms $\varphi : (C_pX, h_{C_pX}) \to (\mathbb{R}, h_{\mathbb{R}})$ are precisely the morphisms $\hat{x}$, where $x \in X$, that is, the point evaluation map.*

Proof. Suppose $\varphi = \hat{x}$, for some $x \in X$. Let $y \in MC_pX$. Take $h_X = C_p\eta_{C_pX}$ and $h_{\mathbb{R}} = \hat{1}_{\mathbb{R}}$. Then $\varphi \circ h_X(y) = \hat{x} \circ C_p\eta_{C_pX}(y) = \hat{x}(C_p\eta_{C_pX}(y)) = y \circ \eta_X(x) = y(\hat{x})$. On the other hand, $h_{\mathbb{R}} \circ M(\varphi)(y) = h_{\mathbb{R}}(M(\varphi)(y)) = \hat{1}_{\mathbb{R}}(M(\varphi)(y)) = M(\varphi)(y)(\mathbf{1}_{\mathbb{R}}) = y(\mathbf{1}_{\mathbb{R}} \circ \varphi) = y(\varphi) = y(\hat{x})$. Therefore, $\varphi \circ h_X = h_{\mathbb{R}} \circ M(\varphi)$ and thus $\varphi = \hat{x}$ is an algebra morphism. The converse follows from Proposition 3.7. □

Proposition 3.9. *The algebra morphisms $\varphi : (C_pX, h_{C_pX}) \to (C_pY, h_{C_pY})$ are the maps $C_p(f)$, where $f : Y \to X$ is continuous.*

Proof. Suppose that $\varphi : (C_pX, h_{C_pX}) \to (C_pY, h_{C_pY})$ is an algebra map. Given $y \in Y$, $\hat{y} \circ \varphi : (C_pX, h_{C_pX}) \to (\mathbb{R}, h_{\mathbb{R}})$ is an algebra map, since the composition of two algebra maps is an algebra map. Thus the following diagram is commutative:

$$\begin{array}{ccccc} MC_pX & \xrightarrow{M\varphi} & MC_p(Y) & \xrightarrow{M\hat{y}} & M\mathbb{R} \\ \Big\downarrow{\scriptstyle h_{C_pX}} & & \Big\downarrow{\scriptstyle h_{C_pY}} & & \Big\downarrow{\scriptstyle h_{\mathbb{R}}} \\ C_pX & \xrightarrow[\varphi]{} & C_pY & \xrightarrow[\hat{y}]{} & \mathbb{R} \end{array} \tag{3.3}$$

By Theorem 3.8, $\hat{y} \circ \varphi = \hat{x}$ for some $x \in X$. Put $x = f(y)$. Thus f maps Y into X. Since X has the initial topology induced by η_X, f will be continuous if $\eta_X \circ f$ is continuous. Now $\eta_X \circ f(y) = \hat{x} = \hat{y} \circ \varphi = C\varphi(\eta_Y(y))$. Thus $\eta_X \circ f = C\varphi \circ \eta_Y$, so that $\eta_X \circ f$ is continuous, hence f is continuous, as required. It remains to prove that $\varphi = Cf$. Since the functions $\hat{y}$ distinguish the points of C_pY, it suffices to prove that $\hat{y} \circ \varphi = \hat{y} \circ Cf$ for every $y \in Y$. Now $\hat{y}(Cf(g)) = \hat{y}(g \circ f) = g \circ f(y) = g(f(y)) = g(x)$. Also $\hat{y}(\varphi(g)) = \hat{y} \circ \varphi(g) = \hat{x}(g) = g(x)$. Hence $\hat{y} \circ \varphi = \hat{y} \circ Cf$ for all $y \in Y$, so that $\varphi = Cf$.

Conversely suppose the morphism $\varphi : (C_pX, h_{C_pX}) \to (C_pY, h_{C_pY})$ is such that $\varphi = Cf$. Then by Proposition 3.9, φ is an algebra morphism. □

Proposition 3.10. *The map $h_{C_pX} : MC_pX \to C_pX$ preserves the ring structure of the function spaces, operations being defined pointwise.*

Proof. Let $\varphi, \psi \in MC_pX$, so that $\varphi, \psi : MX \to \mathbb{R}$. The maps $\varphi + \psi$, $\varphi \cdot \psi$, and $t\varphi$ (where $t \in \mathbb{R}$) are both defined pointwise, so that $(\varphi + \psi)(\lambda) = \varphi(\lambda) + \psi(\lambda)$, $\varphi \cdot \psi(\lambda) = \varphi(\lambda) \cdot \psi(\lambda)$, and $(t\varphi)(\lambda) = t\varphi(\lambda)$, for all $\lambda \in C_pX$. Now $h_{C_pX}(\varphi) = C\eta_X(\varphi) = \varphi \circ \eta_X$, hence $h_{C_pX}(\varphi + \psi) = (\varphi + \psi) \circ \eta_X$. Thus

$$\begin{aligned} (\varphi + \psi) \circ \eta_X(x) &= (\varphi + \psi)(\eta_X(x)) = \varphi(\eta_X(x)) + \psi(\eta_X(x)) \\ &= h_{C_pX}(\varphi)(x) + h_{C_pX}(\psi)(x) = (h_{C_pX}(\varphi) + h_{C_pX}(\psi))(x). \end{aligned} \tag{3.4}$$

Since this holds for every $x \in X$, we have $h_{C_pX}(\varphi + \psi) = h_{C_pX}(\varphi) + h_{C_pX}(\psi)$.

The proof that $h_{C_pX}(\varphi \cdot \psi) = h_{C_pX}(\varphi) \cdot h_{C_pX}(\psi)$ is similar. We also have $h_{C_pX}(t\varphi) = th_{C_pX}(\varphi)$, where t is a scalar. □

Proposition 3.11. *For any $f : Y \to X$, the map $C_pf : C_pX \to C_pY$ preserves the ring structure.*

Proof. Since C_pf acts by composition on the right, the result is clear. We will verify one case only: $C_pf(\varphi+\psi) = C_pf(\varphi)+C_pf(\psi)$. Then

$$\begin{aligned} C_pf(\varphi+\psi)(y) &= (\varphi+\psi)\big(f(y)\big) = \varphi\big(f(y)\big)+\psi\big(f(y)\big) \\ &= C_pf(\varphi)(y)+C_pf(\psi)(y) = \big(C_pf(\varphi)+C_pf(\psi)\big)(y). \end{aligned} \tag{3.5}$$

Since the equality holds for every $y \in Y$, $C_pf(\varphi+\psi) = C_pf(\varphi)+C_pf(\psi)$. □

Problem 3.12. Characterize fully the Eilenberg-Moore category of M-algebras.

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Koena R. Nailana: Department of Mathematics, Applied Mathematics and Astronomy, University of South Africa, P.O. Box 392, Pretoria 0003, South Africa

E-mail address: nailakr@unisa.ac.za