IJMMS 31:5 (2002) 291–299
PII. S016117120210915X
http://ijmms.hindawi.com
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ON THE TIME-DEPENDENT PARABOLIC WAVE EQUATION

ARTHUR D. GORMAN

Received 12 September 2001

One approach to the study of wave propagation in a restricted domain is to approximate the reduced Helmholtz equation by a parabolic wave equation. Here we consider wave propagation in a restricted domain modelled by a parabolic wave equation whose properties vary both in space and in time. We develop a Wentzel-Kramers-Brillouin (WKB) formalism to obtain the asymptotic solution in noncaustic regions and modify the Lagrange manifold formalism to obtain the asymptotic solution near caustics. Associated wave phenomena are also considered.

2000 Mathematics Subject Classification: 34E20.

1. Introduction. One of the most fundamental equations in applied mathematics and mathematical physics is the wave equation

$$\frac{\partial^2\Psi(\overline{r},t)}{\partial x^2}+\frac{\partial^2\Psi(\overline{r},t)}{\partial y^2}+\frac{\partial^2\Psi(\overline{r},t)}{\partial z^2}=\varepsilon(\overline{r},t)\frac{\partial^2\Psi(\overline{r},t)}{\partial t^2}. \tag{1.1}$$

In this equation $\overline{r}=(x,y,z)$ represents the spatial coordinates, t is the time, $\Psi(\overline{r},t)$ is the wave function, and $\varepsilon(\overline{r},t)$ is a continuous function that characterizes the spatial and temporal properties of the propagation medium. When the medium may be modelled as time-invariant, $\varepsilon(\overline{r},t)\to\varepsilon(\overline{r})$. Further, if the propagation is monochromatic and time-harmonic, (1.1) becomes the reduced Helmholtz equation

$$\frac{\partial^2\Psi(\overline{r})}{\partial x^2}+\frac{\partial^2\Psi(\overline{r})}{\partial y^2}+\frac{\partial^2\Psi(\overline{r})}{\partial z^2}+\tau^2\varepsilon(\overline{r})\Psi(\overline{r})=0, \tag{1.2}$$

where τ is a frequency-related parameter.

Because no general approach exists to solve the reduced Helmholtz equation exactly, often situation-specific techniques are employed to determine approximate solutions. One such technique, originally developed by Leontovich and Fock [11] for ionospheric propagation, is the parabolic approximation. In the parabolic approximation, it is assumed that the propagation is restricted to a narrow cone of angles in a particular direction; for Leontovich and Fock, the propagation was directed along the surface of the earth. Later Kravtsov [10] formalized their approach within an asymptotic Wentzel-Kramers-Brillouin (WKB) structure. In 1973, Tappert and Hardin [14] adapted the parabolic approximation for use in underwater acoustics, and subsequently DeSanto [5] examined the accuracy of the approach in the context of underwater acoustics. More recently, the parabolic approximation has been employed in varied geophysical contexts from meteorology to seismology (see [3]). In some atmospheric models, however, the time variation of the medium is significant and cannot be neglected [15]; consequently, a development based on (1.1) is necessary.

Our purpose here is to provide a parabolic approximation formalism for the full wave equation analogous to that provided by Kravtsov for the reduced Helmholtz equation. As in Kravtsov's development, the basis for this formalism is the multidimensional asymptotic WKB structure. Both caustic and noncaustic regions are considered. Further, some wave phenomena usually associated with the full asymptotic wave equation are considered in the context of the parabolic wave equation for both caustic and noncaustic regions.

2. Formalism. Although the parabolic approximation allows the properties of the medium to vary in each of the three dimensions, for a better comparison with Leontovich and Fock and with Kravtsov, the specific form of the wave equation we consider is

$$\frac{\partial^2\Psi(\overline{r},t)}{\partial x^2}+\frac{\partial^2\Psi(\overline{r},t)}{\partial y^2}+\frac{\partial^2\Psi(\overline{r},t)}{\partial z^2}=\varepsilon(x,y,t)\frac{\partial^2\Psi(\overline{r},t)}{\partial t^2}, \tag{2.1}$$

that is, the medium profile ε is cyclic in one variable. To parallel Kravtsov, we assume a solution of the form

$$\Psi(\overline{r},t)=W(\overline{r},t)e^{i\tau z}. \tag{2.2}$$

Then, substituting (2.2) into (2.1) leads to

$$\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}+\frac{\partial^2 W}{\partial z^2}+2i\tau\frac{\partial W}{\partial z}+(i\tau)^2W=\varepsilon(x,y,t)\frac{\partial^2 W}{\partial t^2}. \tag{2.3}$$

Next, the principal assumption of the parabolic approximation is applied, namely, that propagation is restricted to a narrow region in a given direction, taken here to be along the z-axis. This restriction can also serve as justification for the cyclicity of ε in the z-direction, that is, the properties of the medium are constant in the z-direction over the considered range of propagation. Mathematically, this is modelled as

$$\left|\frac{\partial^2 W}{\partial z^2}\right|\ll\left|2i\tau\frac{\partial W}{\partial z}\right|, \tag{2.4}$$

turning (2.3) into

$$\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}+2i\tau\frac{\partial W}{\partial z}+(i\tau)^2W=\varepsilon(x,y,t)\frac{\partial^2 W}{\partial t^2}. \tag{2.5}$$

Then analogous to Kravtsov, away from caustics, we seek an asymptotic solution of the form

$$W(x,y,z,t)\sim\sum_{n=0}^{\infty}A_n(x,y,z,t)(i\tau)^{-n}\exp\{i\tau\phi(x,y,z,t)\}=O(\tau^{-n}), \tag{2.6}$$

where $\phi(x,y,z,t)$ is regarded as a phase and the A_n's are amplitudes. Then, substituting (2.6) into (2.5) and regrouping by powers of $i\tau$ leads to

$$\begin{aligned}\sum_{n=0}\Bigg\{&(i\tau)^2\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+2\frac{\partial\phi}{\partial z}+1-\varepsilon(x,y,t)\left(\frac{\partial\phi}{\partial t}\right)^2\right]A_n\\&+(i\tau)\bigg[2\frac{\partial\phi}{\partial x}\frac{\partial A_n}{\partial x}+2\frac{\partial\phi}{\partial y}\frac{\partial A_n}{\partial y}+2\frac{\partial A_n}{\partial z}-2\varepsilon(x,y,t)\frac{\partial\phi}{\partial t}\frac{\partial A_n}{\partial t}\\&\qquad+\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}-\varepsilon(x,y,t)\frac{\partial^2\phi}{\partial t^2}\right)A_n\bigg]\\&+(i\tau)^0\left[\frac{\partial^2A_n}{\partial x^2}+\frac{\partial^2A_n}{\partial y^2}-\varepsilon(x,y,t)\frac{\partial^2A_n}{\partial t^2}\right]\Bigg\}\exp\{i\tau\phi\}\sim 0.\end{aligned}\tag{2.7}$$

Next, introducing the wave vectors and frequency

$$\overline{p}=\nabla\phi,\qquad \omega=-\frac{\partial\phi}{\partial t},\tag{2.8}$$

respectively, turns the coefficient of the $(i\tau)^2$ into an equation analogous to the eikonal equation of geometrical optics

$$p_x^2+p_y^2+2p_z+1-\varepsilon(x,y,t)\omega^2=0,\tag{2.9}$$

which may be regarded as a Hamiltonian

$$H=p_x^2+p_y^2+2p_z+1-\varepsilon(x,y,t)\omega^2.\tag{2.10}$$

Then, the phase ϕ may be determined from the Hamilton equations

$$\begin{aligned}\frac{d\overline{r}}{d\gamma}&=\nabla_pH, &\qquad \frac{d\overline{p}}{d\gamma}&=-\nabla_rH,\\ \frac{dt}{d\gamma}&=-\frac{\partial H}{\partial\omega}, &\qquad \frac{d\omega}{d\gamma}&=\frac{\partial H}{\partial t},\end{aligned}\tag{2.11}$$

where γ is a ray-path parameter. Specifically, the Hamilton equations lead to the trajectories

$$\begin{aligned}\overline{r}&=\overline{r}(\gamma,\overline{\sigma}), &\qquad \overline{p}&=\overline{p}(\gamma,\overline{\sigma}),\\ t&=t(\gamma,\overline{\sigma}), &\qquad \omega&=\omega(\gamma,\overline{\sigma}),\end{aligned}\tag{2.12}$$

where $\overline{\sigma}$ is a parametrized initial condition. Then, inversion of the time and coordinate transformations yields $\gamma=\gamma(t,\overline{r})$ and $\overline{\sigma}=\overline{\sigma}(t,\overline{r})$. After substituting for γ and $\overline{\sigma}$ in the wave vector and frequency equations, an integration along the trajectories determines the phase

$$\phi(x,y,z,t)=\int\overline{p}\cdot d\overline{r}-\omega\,dt,\tag{2.13}$$

(see [6, pages 125–128]). Upon determination of the phase, the transport equation for the amplitudes proceeds from the $i\tau$ and $(i\tau)^0$ terms in (2.7). Specifically, introducing the wavevectors and frequency, (2.8), determines the transport equation

$$\frac{dA_n}{dt}+\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}-\varepsilon(x,y,t)\frac{\partial^2\phi}{\partial t^2}\right)A_n+\frac{\partial^2 A_{n-1}}{\partial x^2}+\frac{\partial^2 A_{n-1}}{\partial y^2}-\varepsilon(x,y,t)\frac{\partial^2 A_{n-1}}{\partial t^2}=0, \tag{2.14}$$

see [9, equation (14)] for the corresponding equation in autonomous media. With the phase and transport equation determined, the complete off-caustic solution now follows from (2.2).

This procedure applies at most field points. At those space-time points where the coordinate map becomes singular, that is, on the caustic curve where

$$\det\left(\frac{\partial(\overline{r})}{\partial(\overline{\mu})}\right)=0, \tag{2.15}$$

with $\overline{\mu}=(y,\overline{\sigma})$, however, the procedure predicts unbounded wave amplitudes. A related approach that is valid on the caustic curve is the Lagrange manifold formalism developed by Maslov [12] and Arnol'd [2]. This approach has been adapted to the parabolic approximation to the reduced wave equation [9] and to wave propagation in time-dependent media [1, 7]. Here we adapt the procedure to consider caustics associated with the time-dependent parabolic wave equation.

Near caustics, we assume that (2.5), that is, the parabolic wave equation, has a solution of the form

$$W(x,y,z,t)-\int A(\overline{r},\overline{p},t,\tau)\exp\{i\tau\phi(\overline{r},\overline{p},t)\}d\overline{p}_\perp=O(\tau^{-\infty}), \tag{2.16}$$

where $\overline{p}_\perp=(p_x,p_y)$. The amplitude $A(\overline{r},\overline{p},t,\tau)=\sum_{n=0}A_n(\overline{r},\overline{p},t)(i\tau)^{-n}$ and its derivatives are assumed bounded and the phase $\phi(\overline{r},\overline{p},t)$ has the form

$$\phi(\overline{r},\overline{p},t)=xp_x+yp_y+zp_z-S(p_x,p_y,t). \tag{2.17}$$

Then, carrying the differentiation in (2.1) across the integral in (2.17) and introducing the wavevectors and frequency from (2.8) leads to

$$\begin{aligned}\int d\overline{p}\Big\{&(i\tau)^2\big[p_x^2+p_y^2+2p_z+1-\varepsilon(x,y,t)\omega^2\big]A_n\\&+(i\tau)\left[2p_x\frac{\partial A_n}{\partial x}+2p_y\frac{\partial A_n}{\partial y}+2\frac{\partial A_n}{\partial z}+2\omega\frac{\partial A_n}{\partial t}\varepsilon(x,y,t)\right]\\&+(i\tau)^0\left[\frac{\partial^2 A_n}{\partial x^2}+\frac{\partial^2 A_n}{\partial y^2}-\varepsilon(x,y,t)\frac{\partial^2 A_n}{\partial t^2}\right]\exp\{i\tau\phi\}\Big\}\sim O(\tau^{-n}).\end{aligned} \tag{2.18}$$

The coefficient of the $(i\tau)^2$ term is Maslov's Hamiltonian which is identical with (2.10),

$$H=p_x^2+p_y^2+2p_z+1-\varepsilon(x,y,t)\omega^2. \tag{2.19}$$

The integral in (2.18) may be evaluated at any caustic point $(\overline{r},t)$ using the stationary phase technique

$$\nabla_{p_\perp}\phi=0,\quad\text{where }\nabla_{p_\perp}=\hat{i}\frac{\partial}{\partial p_x}+\hat{j}\frac{\partial}{\partial p_y}, \tag{2.20}$$

which turns the Hamiltonian on the caustic into the off-caustic eikonal equation of geometrical optics and determines the time-parametrized Lagrange manifold

$$\overline{r}_\perp = \nabla_p S(\overline{p}_\perp, t), \tag{2.21}$$

where $\overline{r}_\perp = (x, y)$. A Lagrange manifold may be regarded as a coordinate transformation between configuration space and wavevector space specified by the generating function $S(\overline{p}_\perp, t)$. To determine $S(\overline{p}_\perp, t)$, we again employ the Hamilton equations (2.11) for $\overline{r}_\perp$ and $\overline{p}_\perp$ to find the trajectories (maps)

$$\begin{aligned} \overline{r}_\perp &= \overline{r}_\perp(\gamma, \overline{\sigma}), \qquad & \overline{p}_\perp &= \overline{p}_\perp(\gamma, \overline{\sigma}), \\ t &= t(\gamma, \overline{\sigma}), & \omega &= \omega(\gamma, \overline{\sigma}), \end{aligned} \tag{2.22}$$

where $\overline{\sigma}$ is again a parametrized initial condition. Next, the map $(\gamma, \overline{\sigma}) \to (t, \overline{p}_\perp)$ is inverted to obtain γ and $\overline{\sigma}$ as functions of t and $\overline{p}_\perp$. Then, substituting into the coordinate space map determines the Lagrange manifold explicitly

$$\overline{r}_\perp = \overline{r}_\perp(\gamma(\overline{p}_\perp, t), \overline{\sigma}(\overline{p}_\perp, t)) = \nabla_{p_\perp} S(\overline{p}_\perp, t), \tag{2.23}$$

where time appears as a parameter [1, 7]. Finally, an integration along the trajectories gives

$$S(\overline{p}_\perp, t) = \int_{\overline{p}_0}^{\overline{p}} \overline{r}_\perp \cdot d\overline{p}_\perp, \tag{2.24}$$

and hence the phase

$$\phi(\overline{r}, \overline{p}, t) = \overline{r} \cdot \overline{p} - S(\overline{p}_\perp, t). \tag{2.25}$$

To determine a transport equation for the amplitudes, we form the Taylor expansion of the Hamiltonian near the Lagrange manifold

$$\begin{aligned} p_x^2 + p_y^2 + 2p_z + 1 - \varepsilon(x, y, t)\omega^2 &= p_x^2 + p_y^2 + 2p_z + 1 - \varepsilon\left(\frac{\partial S}{\partial p_x}, \frac{\partial S}{\partial p_y}, t\right)\omega^2 \\ &\quad + (\overline{r}_\perp - \nabla_{p_\perp} S) \cdot \overline{D}_\perp = (\overline{r}_\perp - \nabla_{p_\perp} S) \cdot \overline{D}_\perp, \end{aligned} \tag{2.26}$$

where

$$\overline{D}_\perp = \int_0^1 \nabla_{r_\perp} H[\xi(\overline{r}_\perp - \nabla_{p_\perp} S) + (\overline{r}_\perp - \nabla_{p_\perp} S), \overline{p}, \omega, t]\, d\xi, \tag{2.27}$$

that is, the remainder of the Taylor series. Then, substituting (2.26) into the integral in (2.18) and performing a partial integration leads to

$$\begin{aligned} \int d\overline{p}_\perp \Big\{ (i\tau) \Big[\overline{p}_\perp \cdot \nabla_{r_\perp} A + \omega(x, y, t) \frac{\partial A}{\partial t} + (1) \frac{\partial A}{\partial z} - \nabla_{p_\perp} \cdot \overline{D}_\perp A - \overline{D}_\perp \cdot \nabla_{p_\perp} A \Big] \\ + (i\tau)^0 \Big[\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} - \varepsilon(x, y, t) \frac{\partial^2 A}{\partial t^2} \Big] \Big\} \exp\{i\tau\phi\} \sim O(\tau^{-n}). \end{aligned} \tag{2.28}$$

Finally, introducing the non-Hamiltonian flow

$$\begin{aligned} \dot{\overline{r}}_\perp &= \overline{p}_\perp, & \dot{\overline{p}}_\perp &= -\overline{D}_\perp, \\ \dot{z} &= 1, & \dot{\omega} &= t, \end{aligned} \tag{2.29}$$

where $\cdot \triangleq d/dy$, into the integral leads to a transport equation for the amplitudes

$$\frac{dA_n}{dy} - \nabla_{p_\perp} \cdot \overline{D} A_n + \frac{\partial^2 A_{n-1}}{\partial x^2} + \frac{\partial^2 A_{n-1}}{\partial y^2} - \varepsilon(x,y,t)\frac{\partial^2 A_{n-1}}{\partial t^2} = 0. \tag{2.30}$$

With the phase and transport equation determined, the complete solution on the caustic curve now follows from (2.2). Because explicit examples of the application of the parabolic approximation to the reduced Helmholtz equation [7] and of related propagation problems in time-varying media have already been detailed elsewhere [1, 7], for brevity, we do not include others here.

3. Analysis. The parabolic approximation prevents a Hamiltonian analysis of the ray-path in the direction of the approximation, here the z direction. In those directions unaffected by the approximation, however, the usual analysis stemming from the Hamilton equations yields information corresponding to that obtained without the approximation.

Since the WKB and Lagrange manifold algorithms determines identical Hamiltonians and eikonal equations on and off the caustic, the dispersion relation both on and off the caustic becomes

$$\omega^2 = \frac{p_x^2 + p_y^2 + p_z + 1}{\varepsilon(x,y,t)}. \tag{3.1}$$

Consequently, both off and on the caustic the phase velocities in the x and y directions are

$$\begin{aligned} v_{px} &= \frac{\omega}{p_x} = \sqrt{\frac{p_x^2 + p_y^2 + 2p_z + 1}{\varepsilon(x,y,t)p_x^2}}, \\ v_{py} &= \frac{\omega}{p_y} = \sqrt{\frac{p_x^2 + p_y^2 + 2p_z + 1}{\varepsilon(x,y,t)p_y^2}}. \end{aligned} \tag{3.2}$$

Further, from the dispersion relation, the group velocities are given by

$$\begin{aligned} C_{gx} &= \frac{\partial \omega}{\partial p_x} = \frac{p_x}{\varepsilon(x,y,t)\omega}, \\ C_{gy} &= \frac{\partial \omega}{\partial p_y} = \frac{p_y}{\varepsilon(x,y,t)\omega}. \end{aligned} \tag{3.3}$$

We also note that C_{gx} and C_{gy} may also be determined from the Hamilton equations if we replace the ray-path parameter y with t (see [4]). In the x and y directions, the Hamilton equations then become as follows:

$$\frac{dx}{dt} = \frac{\partial H/\partial p_x}{-\partial H/\partial \omega} = C_{gx}, \tag{3.4a}$$

$$\frac{dy}{dt} = \frac{\partial H/\partial p_y}{-\partial H/\partial \omega} = C_{gy}, \tag{3.4b}$$

$$\frac{dt}{dt} = \frac{-\partial H/\partial \omega}{-\partial H/\partial \omega} = 1, \tag{3.4c}$$

$$\frac{dp_x}{dt} = -\frac{\partial H/\partial x}{-\partial H/\partial \omega} = \omega^2 \frac{\partial \varepsilon}{\partial x}, \tag{3.4d}$$

$$\frac{dp_y}{dt} = -\frac{\partial H/\partial y}{-\partial H/\partial \omega} = \omega^2 \frac{\partial \varepsilon}{\partial y}, \tag{3.4e}$$

$$\frac{d\omega}{dt} = \frac{\partial H/\partial t}{-\partial H/\partial \omega} = \omega^2 \frac{\partial \varepsilon}{\partial t}. \tag{3.4f}$$

These equations apply both on and off the caustic curve. Consequently, in the directions in which the parabolic approximation is not applied, we observe the following phenomenological prediction apply both on and off the caustic curve:

(1) the direction of propagation coincides with the group velocity, (3.4a) and (3.4b);

(2) any change in the local momentum (wavevector) in a given direction must be the result of a spatial inhomogeneity acting in that direction. If the medium is cyclic in a given direction, the momentum (wavevector) is conserved in that direction, (3.4d) and (3.4e);

(3) any change in the frequency must be the result of a temporal inhomogeneity in the medium. If the medium is autonomous, then the frequency is conserved, (3.4f).

With regard to the location of the caustic curve, we note that the Lagrange manifold approach allows the determination of the caustic directly from the phase. For any fixed value of time, t, setting the Hessian determinant of the phase to zero, that is,

$$\det\left\{\frac{\partial^2 \phi}{\partial p_x \partial p_y}\right\} = 0, \tag{3.5}$$

yields sets of ordered pairs (p_x, p_y). Substitution of these sets into the Lagrange manifold, (2.22), determines the caustic in coordinate space at a particular time. The time evolution of the caustic proceeds by considering (3.5) for several values of time. Corresponding to each value of time is a different set of ordered pairs which, when substituted into the Lagrange manifold, traces the evolution of the caustic [1, 7].

4. Cylindrical coordinates. Because the parabolic approximation is often used with wave propagation in cylindrical coordinates, for completeness, we include a synopsis of the approach in cylindrical geometry. In cylindrical coordinates, the wave equation becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2} = \varepsilon(r,\theta,z,t)\frac{\partial^2 \psi}{\partial t^2}. \tag{4.1}$$

In (4.1), r is regarded as a horizontal range, θ as an azimuthal angle, and z as a distance (depth or height) measured from the horizontal. In many such applications, r is considered the "primary" direction of the propagation and the spatial inhomogeneity appears in the z-direction, that is, ε is modelled as cyclic in r, see [13] for a thorough exposition.

Analogous to (2.2), we assume a solution of the form

$$\psi(r,\theta,z,t) = r^{-1/2}W(r,\theta,z,t)e^{i\tau r}. \tag{4.2}$$

Substitution of (4.2) into (4.1) yields

$$\frac{\partial^2 W}{\partial r^2} + 2i\tau\frac{\partial W}{\partial r} + \frac{\partial^2 W}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2} + \frac{1}{4r^2}W - (i\tau)^2 W = \varepsilon(z,\theta,t)\frac{\partial^2 W}{\partial t^2}. \tag{4.3}$$

In cylindrical coordinates the parabolic approximation, analogous to (2.4), is modelled as

$$\left|\frac{\partial^2 W}{\partial r^2}\right| \ll \left|2i\tau\frac{\partial W}{\partial r}\right|. \tag{4.4}$$

Further, for ranges far from the source, the term $(1/4r^2)W$ may be neglected. Consequently, the analysis begins with an equation of the form

$$2i\tau\frac{\partial W}{\partial r} + \frac{\partial^2 W}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2} - (i\tau)^2 W = \varepsilon(z,\theta,t)\frac{\partial^2 W}{\partial t^2}. \tag{4.5}$$

Often, some situation-specific assumptions are employed to simplify this equation further. At great ranges from a source the curvature of the wave front can be neglected, allowing the variable change $r\,d\theta = dy$, transforming (4.5) into

$$2i\tau\frac{\partial W}{\partial r} + \frac{\partial^2 W}{\partial z^2} + \frac{\partial^2 W}{\partial y^2} - \tau^2 W = \varepsilon(z,t)\frac{\partial^2 W}{\partial t^2}, \tag{4.6}$$

where the azimuthal variation of the medium is assumed small compared to the variations in depth/height and time. Equation (4.6) is of the same form as (2.5); consequently, the algorithm above applies directly.

Another simplification of (4.5) arises from the assumption of azimuthal symmetry leading to

$$2i\tau\frac{\partial W}{\partial r} + \frac{\partial^2 W}{\partial z^2} - (i\tau)^2 W = \varepsilon(z,t)\frac{\partial^2 W}{\partial t^2}. \tag{4.7}$$

With the specification of a particular r and t coordinate pair, this becomes essentially a one-dimensional problem, see [13] for a thorough analysis for the related reduced Helmholtz equation and [8] for a simplified algorithm applicable at caustics.

Acknowledgment. The counsel and encouragement of Gail F. Angell is gratefully acknowledged.

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Arthur D. Gorman: Department of Mathematics, Lafayette College, Easton, PA 18042, USA

E-mail address: gormana@lafayette.edu