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ON THE UNIVALENCY FOR CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS INVOLVING RUSCHEWEYH DERIVATIVES

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Let H be the class of functions $f(z)$ of the form $f(z)=z+\sum_{k=2}^{+\infty}a_kz^k$, which are analytic in the unit disk $U=\{z;\ |z|<1\}$. In this paper, we introduce a new subclass $B_\lambda(\mu,\alpha,\rho)$ of H and study its inclusion relations, the condition of univalency, and covering theorem. The results obtained include the related results of some authors as their special case. We also get some new results.

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1. Introduction. Let H be the class of functions of the form

$$f(z)=z+\sum_{k=2}^{+\infty}a_kz^k \tag{1.1}$$

which are analytic in the unit disk $U=\{z;\ |z|<1\}$. Let S be the subclass of H consisting of univalent functions.

For the function $f(z)=\sum_{k=1}^{+\infty}a_kz^k$ and $g(z)=\sum_{k=1}^{+\infty}b_kz^k$, let $(f*g)(z)$ denote the Hadamard product or convolution of $f(z)$ and $g(z)$, defined by

$$(f*g)(z)=\sum_{k=1}^{+\infty}a_kb_kz^k. \tag{1.2}$$

Now define the function $\phi(a,c;z)$ by

$$\phi(a,c;z)=\sum_{k=0}^{+\infty}\frac{(a)_k}{(c)_k}z^{k+1},\quad (c\neq 0,-1,-2,\dots,\ z\in U), \tag{1.3}$$

where

$$(\lambda)_k=\frac{\Gamma(\lambda+k)}{\Gamma(\lambda)}=\begin{cases}1, & k=0,\\ \lambda(\lambda+1)\cdots(\lambda+k-1), & k\in N=\{1,2,\dots\}.\end{cases} \tag{1.4}$$

It follows from [4] that

$$z[\phi(c,c+1)]'=c\phi(c+1,c+1)-(c-1)\phi(c,c+1). \tag{1.5}$$

Carlson and Shaffer [2] defined a linear operator $L(a,c)$ on H by using the Hadamard product

$$L(a,c)f = \phi(a,c;z) * f(z), \quad f \in H. \tag{1.6}$$

It is known in [2] that $L(a,c)$ maps H into itself. If $a \neq 0,-1,-2,\dots$, then $L(a,c)$ has a continuous inverse $L(c,a)$. Clearly, $L(a,a)$ is the unit operator. Also, if $c > a > 0$, then $L(a,c)$ has the integral representation

$$L(a,c)f(z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 u^{a-2}(1-u)^{c-a-1} f(uz)\,du. \tag{1.7}$$

Ruscheweyh [7] introduced an operator $D^{\lambda}: H \to H$ defined by the Hadamard product or convolution

$$D^{\lambda} f(z) = \frac{z}{(1-z)^{\lambda+1}} * f(z), \quad (\lambda > -1,\ z \in U), \tag{1.8}$$

which implies that

$$\begin{gathered} D^n f(z) = \frac{z\left(z^{n-1} f(z)\right)^{(n)}}{n!}, \quad (n \in N_0 = \{0,1,2,\dots\}), \\ D^{\lambda} f(z) = L(\lambda+1,1) f(z). \end{gathered} \tag{1.9}$$

Assume $\alpha > 0$, $\mu > 0$, $\lambda > -1$, $\rho < 1$, a function $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ is said to be in the class P_ρ if and only if $p(z)$ is analytic in the unit disk U and $\operatorname{Re} p(z) > \rho$, $z \in U$; a function $f(z) \in H$ is said to be in the class $B_\lambda(\mu,\alpha,\rho)$ if and only if it satisfies

$$\operatorname{Re}\left[(1-\mu)\left(\frac{D^{\lambda} f(z)}{z}\right)^{\alpha} + \mu\left(D^{\lambda} f(z)\right)'\left(\frac{D^{\lambda} f(z)}{z}\right)^{\alpha-1}\right] > \rho, \quad z \in U, \tag{1.10}$$

where the power are understood as principle values. Below we apply this agreement. It is obvious that the subclass $B_0(1,\alpha,0)$ is the subclass of Bazilevič functions, which is the subclass of univalent functions S, let $B(\alpha,\rho) \equiv B_0(1,\alpha,\rho)$. The subclass $B_0(1,\alpha,\rho)$ $(0 \le \rho < 1)$ has been studied by Bazilevič [1], Singh [8], respectively. $B_0(0,\alpha,\rho)$ $(\rho < 1)$ has been studied by Liu [5]. The subclass $B_0(\lambda,1,\rho)$ $(0 \le \rho < 1)$ has been studied by Chichra [3], Ding et al. [4], respectively.

In this paper, we study the properties of $B_\lambda(\mu,\alpha,\rho)$. The results obtained generalize the related works of some authors. We also obtained some new univalent criterions.

2. Some lemmas

Lemma 2.1 [4]. *Let $\rho < 1$, $0 < u < 1$, $F(z) \in P_\rho$, then for $|z| = r < 1$,*

$$\operatorname{Re}\left[F(z) - F(uz)\right] \ge \frac{2(1-\rho)(u-1)r}{(1+r)(1+ur)}, \tag{2.1}$$

and the inequality is sharp.

Lemma 2.2. *Let $c > 0$, $\mu > 0$, $\rho < 1$, $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ be analytic in U. If*

$$\operatorname{Re}\left[p(z) + c\mu z p'(z)\right] > \rho, \quad z \in U, \tag{2.2}$$

then for $|z| = r < 1$,

$$\begin{aligned}
\operatorname{Re}\left[p(z) + czp'(z)\right] &\geq 2\rho - 1 + \frac{2(1-\rho)}{\mu(1+r)} + 2(1-\rho)\left(1 - \frac{1}{\mu}\right)\frac{1}{c\mu}\int_0^1 \frac{u^{1/c\mu - 1}}{1+ur}\,du, \\
\operatorname{Re}\left[p(z) + czp'(z)\right] &\geq 2\rho - 1 + \frac{1-\rho}{\mu} + 2(1-\rho)\left(1 - \frac{1}{\mu}\right)\frac{1}{c\mu}\int_0^1 \frac{u^{1/c\mu - 1}}{1+u}\,du,
\end{aligned} \tag{2.3}$$

and these results are sharp.

Proof. Set $F(z) = p(z) + c\mu z p'(z)$, then it follows from (2.2) that $F(z) \in P_\rho$, and

$$zF(z) = (1 - c\mu)\left[zp(z)\right] + c\mu z\left[zp(z)\right]' = L\left(\frac{1}{c\mu} + 1, \frac{1}{c\mu}\right)\left[zp(z)\right] \tag{2.4}$$

that is,

$$zp(z) = L\left(\frac{1}{c\mu}, \frac{1}{c\mu} + 1\right)\left[zF(z)\right] = \frac{1}{c\mu}\int_0^1 u^{1/c\mu - 1} zF(uz)\,du. \tag{2.5}$$

Let $b = 1/c\mu$, then

$$p(z) = b\int_0^1 u^{b-1} F(uz)\,du. \tag{2.6}$$

According to (1.5) and (2.5), we get

$$\begin{aligned}
z\left[zp(z)\right]' &= \left[z\left(\phi(b, b+1; z)\right)'\right] * \left[zF(z)\right] \\
&= bL(b+1, b+1)\left[zF(z)\right] - (b-1)L(b, b+1)\left[zF(z)\right] \\
&= bzF(z) - b(b-1)z\int_0^1 u^{b-1} F(uz)\,du.
\end{aligned} \tag{2.7}$$

On the other hand, we have

$$\left[zp(z)\right]' = p(z) + zp'(z). \tag{2.8}$$

Thus

$$\begin{aligned}
&p(z) + czp'(z) \\
&\quad = (1-c)p(z) + c\left[zp(z)\right]' \\
&\quad = bcF(z) + b(1-c)\int_0^1 u^{b-1} F(uz)\,du - bc(b-1)z\int_0^1 u^{b-1} F(uz)\,du \\
&\quad = bcF(z) + b(1-bc)\int_0^1 u^{b-1} F(uz)\,du.
\end{aligned} \tag{2.9}$$

If $\mu > 1$, then $0 < bc = 1/\mu < 1$, and

$$\begin{aligned}\operatorname{Re}[p(z)+czp'(z)] &= bc\operatorname{Re}[F(z)]+b(1-bc)\int_0^1 u^{b-1}\operatorname{Re}[F(uz)]\,du\\ &\geq bc\cdot\frac{1-(1-2\rho)r}{1+r}+b(1-bc)\int_0^1 u^{b-1}\frac{1-(1-2\rho)ur}{1+ur}\,du \quad (2.10)\\ &= 2\rho-1+\frac{2(1-\rho)}{\mu(1+r)}+2(1-\rho)\left(1-\frac{1}{\mu}\right)\frac{1}{c\mu}\int_0^1\frac{u^{1/c\mu-1}}{1+ur}\,du.\end{aligned}$$

If $0 < \mu \leq 1$, then $bc = 1/\mu \geq 1$, so that it follows from Lemma 2.1 and (2.9) that

$$\begin{aligned}\operatorname{Re}[p(z)+czp'(z)] &= \operatorname{Re}\left[bcF(z)-b(bc-1)\int_0^1 u^{b-1}F(uz)\,du\right]\\ &= \operatorname{Re}F(z)+b(bc-1)\int_0^1 u^{b-1}\operatorname{Re}[F(z)-F(uz)]\,du\\ &\geq \frac{1-(1-2\rho)r}{1+r}+b(bc-1)\int_0^1 u^{b-1}\frac{2(1-\rho)(u-1)r}{(1+r)(1+ur)}\,du\\ &= 2\rho-1+\frac{2(1-\rho)}{\mu(1+r)}+2(1-\rho)\left(1-\frac{1}{\mu}\right)\frac{1}{c\mu}\int_0^1\frac{u^{1/c\mu-1}}{1+ur}\,du.\end{aligned} \tag{2.11}$$

Since the function

$$2\rho-1+\frac{2(1-\rho)}{\mu(1+r)}+2(1-\rho)\left(1-\frac{1}{\mu}\right)\frac{1}{c\mu}\int_0^1\frac{u^{1/c\mu-1}}{1+ur}\,du \tag{2.12}$$

is decreasing with respect to r, therefore

$$\operatorname{Re}[p(z)+czp'(z)] \geq 2\rho-1+\frac{1-\rho}{\mu}+2(1-\rho)\left(1-\frac{1}{\mu}\right)\frac{1}{c\mu}\int_0^1\frac{u^{1/c\mu-1}}{1+u}\,du. \tag{2.13}$$

Note that

$$p_{\mu,c,\rho}(z) = \frac{1}{c\mu}\int_0^1 u^{1/c\mu-1}\frac{1+(1-2\rho)uz}{1-uz}\,du, \tag{2.14}$$

satisfies (2.2), we obtain that the inequalities (2.3) are sharp. □

Lemma 2.3 [6]. *Let $p(z) = 1+p_1z+\cdots \in P_\rho$, then $|p_k| \leq 2-2\rho$, $k = 1,2,\ldots$.*

3. Main results

Theorem 3.1. *Let $\alpha > 0$, $\beta > 0$, $\lambda \geq 0$, $f(z) \in H$ and*

$$\operatorname{Re}\left[(D^\lambda f(z))'\left(\frac{D^\lambda f(z)}{z}\right)^{\alpha-1}\right] > \frac{\lambda}{\beta+\lambda}, \quad z \in U, \tag{3.1}$$

then $f(z)$ is univalent in U, that is, $B_\lambda(1,\alpha,\lambda/(\beta+\lambda)) \subset B(\alpha,0)$.

Proof. Since $\lambda/(\beta+\lambda) \geq 0$, we have

$$B\left(\alpha,\frac{\lambda}{\beta+\lambda}\right) \subset B(\alpha,0). \tag{3.2}$$

Let $T = D^{\lambda} f(z)$, then T is a linear and homeomorphism from $B_{\lambda}(1,\alpha,\lambda/(\beta+\lambda))$ onto $B(\alpha,\lambda/(\beta+\lambda))$, therefore it follows from (3.2) that

$$B_{\lambda}\left(1,\alpha,\frac{\lambda}{\beta+\lambda}\right) = T^{-1}B\left(\alpha,\frac{\lambda}{\beta+\lambda}\right) \subset T^{-1}B(\alpha,0) = B_0(1,\alpha,0) \equiv B(\alpha,0). \tag{3.3}$$

Hence the proof is completed. □

Since $\lim_{\beta\to\infty} \lambda/(\beta+\lambda) = 0$, the following corollary follows from Theorem 3.1.

Corollary 3.2. *Let $\alpha > 0$, $0 < \rho < 1$, $\lambda \geq 0$, then*

$$B_{\lambda}(1,\alpha,\rho) \subset B(\alpha,0) \subset S. \tag{3.4}$$

Theorem 3.3. *Let $\mu_2 \geq \mu_1 > 0$, $1 > \rho_2 \geq \rho_1$, then*

$$B_{\lambda}(\mu_2,\alpha,\rho_2) \subset B_{\lambda}(\mu_1,\alpha,\rho_1). \tag{3.5}$$

Proof. First, it is obvious that

$$B_{\lambda}(\mu_2,\alpha,\rho_2) \subset B_{\lambda}(\mu_2,\alpha,\rho_1). \tag{3.6}$$

Therefore we only need to verify that

$$B_{\lambda}(\mu_2,\alpha,\rho_1) \subset B_{\lambda}(\mu_1,\alpha,\rho_1). \tag{3.7}$$

Let $p(z) = [D^{\lambda} f(z)/z]^{\alpha}$ for $f \in B_{\lambda}(\mu_2,\alpha,\rho_1)$, where the power are understood as principle values, then $p(z) = 1 + (1+\lambda)\alpha a_2 z + \cdots$ is analytic in U and

$$[D^{\lambda} f(z)]^{\alpha} = z^{\alpha} p(z). \tag{3.8}$$

By taking the derivatives in the both sides of (3.8), we obtain

$$(1-\mu_2)\left(\frac{D^{\lambda} f(z)}{z}\right)^{\alpha} + \mu_2 (D^{\lambda} f(z))'\left(\frac{D^{\lambda} f(z)}{z}\right)^{\alpha-1} = p(z) + \frac{\mu_2}{\alpha} z p'(z). \tag{3.9}$$

Since $f \in B_{\lambda}(\mu_2,\alpha,\rho_1)$, we have

$$\operatorname{Re}\left[p(z) + \frac{\mu_2}{\alpha} z p'(z)\right] > \rho_1, \quad z \in U. \tag{3.10}$$

According to Lemma 2.2, we obtain

$$\begin{aligned}\operatorname{Re}\left[p(z) + \frac{\mu_1}{\alpha} z p'(z)\right] &\geq 2\rho_1 - 1 + \frac{2(1-\rho_1)}{\mu(1+r)} + 2(1-\rho_1)\left(1-\frac{1}{\mu}\right)\frac{\alpha}{\mu}\int_0^1 \frac{u^{\alpha/\mu-1}}{1+ur}\,du \\ &\geq 2\rho_1 - 1 + \frac{2(1-\rho_1)}{\mu} + 2(1-\rho_1)\left(1-\frac{1}{\mu}\right)\cdot\frac{1}{2} > \rho_1\end{aligned} \tag{3.11}$$

for $\mu = \mu_2/\mu_1 \geq 1$. Hence it follows from (3.6) and (3.11) that

$$\operatorname{Re}\left[(1-\mu_1)\left(\frac{D^\lambda f(z)}{z}\right)^\alpha + \mu_1 (D^\lambda f(z))'\left(\frac{D^\lambda f(z)}{z}\right)^{\alpha-1}\right] = \operatorname{Re}\left[p(z) + \frac{\mu_1}{\alpha} z p'(z)\right] > \rho_1, \tag{3.12}$$

that is,

$$f \in B_\lambda(\mu_2, \alpha, \rho_1). \tag{3.13}$$

Hence

$$B_\lambda(\mu_2, \alpha, \rho_1) \subset B_\lambda(\mu_1, \alpha, \rho_1). \tag{3.14}$$

□

According to Theorem 3.3 and Corollary 3.2, we have the following corollary.

Corollary 3.4. *Let $\lambda \geq 0$, $\mu \geq 1$, $0 < \rho < 1$, then*

$$B_\lambda(\mu, \alpha, \rho) \subset B_\lambda(1, \alpha, \rho) \subset S. \tag{3.15}$$

Theorem 3.5. *Let $\alpha > 0$, $\lambda \geq 0$, $\mu > 0$, $\rho < 1$. If $f \in B_\lambda(\mu, \alpha, \rho)$, then for $|z| = r < 1$,*

$$\begin{aligned}
&\operatorname{Re}\left[(D^\lambda f(z))'\left(\frac{D^\lambda f(z)}{z}\right)^{\alpha-1}\right] \geq 2\rho - 1 + \frac{2(1-\rho)}{\mu(1+r)} + 2(1-\rho)\left(1-\frac{1}{\mu}\right)\frac{\alpha}{\mu}\int_0^1 \frac{u^{\alpha/\mu-1}}{1+ur}\,du, \\
&\operatorname{Re}\left[(D^\lambda f(z))'\left(\frac{D^\lambda f(z)}{z}\right)^{\alpha-1}\right] \geq 2\rho - 1 + \frac{1-\rho}{\mu} + 2(1-\rho)\left(1-\frac{1}{\mu}\right)\frac{\alpha}{\mu}\int_0^1 \frac{u^{\alpha/\mu-1}}{1+u}\,du,
\end{aligned} \tag{3.16}$$

and these results are sharp.

Proof. Let $p(z) = [D^\lambda f(z)/z]^\alpha$ for $f \in B_\lambda(\mu, \alpha, \rho)$, where the power are understood as principle values, then

$$p(z) = 1 + (1+\lambda)\alpha a_2 z + \cdots \tag{3.17}$$

is analytic in U and

$$[D^\lambda f(z)]^\alpha = z^\alpha p(z). \tag{3.18}$$

By taking the derivatives in the both sides of (3.18), we obtain

$$(1-\mu)\left(\frac{D^\lambda f(z)}{z}\right)^\alpha + \mu (D^\lambda f(z))'\left(\frac{D^\lambda f(z)}{z}\right)^{\alpha-1} = p(z) + \frac{\mu}{\alpha} z p'(z). \tag{3.19}$$

Since $f \in B_\lambda(\mu, \alpha, \rho)$, we have

$$\operatorname{Re}\left[p(z) + \frac{\mu}{\alpha} z p'(z)\right] > \rho, \quad z \in U. \tag{3.20}$$

According to Lemma 2.2, we obtain

$$\operatorname{Re}\left[(D^\lambda f(z))'\left(\frac{D^\lambda f(z)}{z}\right)^{\alpha-1}\right] = \operatorname{Re}\left[p(z)+\frac{1}{\alpha}zp'(z)\right]$$
$$\geq 2\rho-1+\frac{2(1-\rho)}{\mu(1+r)}+2(1-\rho)\left(1-\frac{1}{\mu}\right)\frac{\alpha}{\mu}\int_0^1\frac{u^{\alpha/\mu-1}}{1+ur}du,$$
$$\operatorname{Re}\left[(D^\lambda f(z))'\left(\frac{D^\lambda f(z)}{z}\right)^{\alpha-1}\right] = \operatorname{Re}\left[p(z)+\frac{1}{\alpha}zp'(z)\right]$$
$$\geq 2\rho-1+\frac{1-\rho}{\mu}+2(1-\rho)\left(1-\frac{1}{\mu}\right)\frac{\alpha}{\mu}\int_0^1\frac{u^{\alpha/\mu-1}}{1+u}du. \tag{3.21}$$

Note that,

$$f_{\lambda,\mu,\alpha,\rho}(z) = L(1,\lambda+1)\left\{z\left[\frac{\alpha}{\mu}\int_0^1 u^{\alpha/\mu-1}\frac{1+(1-2\rho)uz}{1-uz}du\right]^{1/\alpha}\right\}\in B_\lambda(\mu,\alpha,\rho), \tag{3.22}$$

we obtain that inequalities (3.16) are sharp. □

Remark 3.6. Setting $\lambda = 0$, $\alpha = 1$ in Theorem 3.5, we get [4, Theorem 1(ii)].

Theorem 3.7. *Let $\alpha > 0$, $\mu > 0$, $\lambda \geq 0$, $\rho_0 \leq \rho < 1$, then $B_\lambda(\mu,\alpha,\rho) \subset B_\lambda(1,\alpha,\rho_1) \subset S$, where*

$$\rho_0 = 1-\frac{1}{2-1/\mu-2(1-1/\mu)(\alpha/\mu)\int_0^1(u^{\alpha/\mu-1}/(1+u))\,du}, \tag{3.23}$$

and the constant ρ_0 cannot be replaced by any smaller one.

Proof. Let $f(z) \in B_\lambda(\mu,\alpha,\rho)$, then it follows from Theorem 3.5 that

$$\operatorname{Re}\left[(D^\lambda f(z))'\left(\frac{D^\lambda f(z)}{z}\right)^{\alpha-1}\right] > \rho_1, \quad z\in U, \tag{3.24}$$

where

$$\begin{aligned}\rho_1 &= 2\rho-1+\frac{1-\rho}{\mu}+2(1-\rho)\left(1-\frac{1}{\mu}\right)\frac{\alpha}{\mu}\int_0^1\frac{u^{\alpha/\mu-1}}{1+u}du\\ &= 1-(1-\rho)\left[2-\frac{1}{\mu}-2\left(1-\frac{1}{\mu}\right)\frac{\alpha}{\mu}\int_0^1\frac{u^{\alpha/\mu-1}}{1+u}du\right].\end{aligned} \tag{3.25}$$

Since

$$\frac{1}{2}<\frac{\alpha}{\mu}\int_0^1\frac{u^{\alpha/\mu-1}}{1+u}du = \frac{1}{2}+\int_0^1\frac{u^{\alpha/\mu}}{(1+u)^2}du < 1, \tag{3.26}$$

so that

$$\max\left\{1,\frac{1}{\mu}\right\} > 2-\frac{1}{\mu}-2\left(1-\frac{1}{\mu}\right)\frac{\alpha}{\mu}\int_0^1\frac{u^{\alpha/\mu-1}}{1+u}du > \min\left\{1,\frac{1}{\mu}\right\} > 0. \tag{3.27}$$

Therefore from $\rho_0 \le \rho < 1$, we have

$$\rho_1 > 1-(1-\rho_0)\left[2-\frac{1}{\mu}-2\left(1-\frac{1}{\mu}\right)\frac{\alpha}{\mu}\int_0^1 \frac{u^{\alpha/\mu-1}}{1+u}\,du\right] = 0. \tag{3.28}$$

Hence it follows from (1.10) and Corollary 3.2 that $f(z) \in B_\lambda(1,\alpha,\rho_1) \subset S$ and $f(z)$ is univalent in U, hence

$$B_\lambda(\mu,\alpha,\rho) \subset B_\lambda(1,\alpha,\rho_1) \subset S \tag{3.29}$$

and the constant ρ_0 cannot be replaced by any smaller one from Theorem 3.5. □

Remark 3.8. Setting $\lambda = 0$, $\alpha = 1$ in Theorem 3.7, we get [4, Theorem 2]; setting $\lambda = 0$, $\mu = 1$ in Theorem 3.7, we get the result of [1].

Setting $\lambda = 0$, $\mu = \alpha > 0$ in Theorem 3.7, we have the following corollary.

Corollary 3.9. *If $f(z) \in H$, and*

$$\operatorname{Re}\left[(1-\alpha)\left(\frac{f(z)}{z}\right)^{\alpha}+\alpha f'(z)\left(\frac{f(z)}{z}\right)^{\alpha-1}\right] > \rho_\alpha = \frac{(\alpha-1)(1-2\ln 2)}{\alpha+(\alpha-1)(1-2\ln 2)}, \quad z \in U, \tag{3.30}$$

then $f(z)$ is univalent in U, and the result is sharp.

Remark 3.10. We note that $\rho_\alpha < 0$ for $\alpha > 1$.

Setting $\mu = \alpha = 3$, $\lambda = 1$ in Theorem 3.7, we have the following corollary.

Corollary 3.11. *If $f(z) \in H$, and*

$$\operatorname{Re}\left\{z\left[(f'(z))^3\right]'+(f'(z))^3\right\} > \frac{2-4\ln 2}{5-4\ln 2} \approx -0.34, \quad z \in U, \tag{3.31}$$

then $f(z)$ is univalent in U.

Theorem 3.12. *Let $f(z) = z+\sum_{k=2}^{+\infty} a_k z^k \in B_\lambda(\mu,\alpha,\rho)$, then*

$$|a_2| \le \frac{2-2\rho}{(1+\lambda)(\alpha+\mu)}, \tag{3.32}$$

and the inequality is sharp, with the extremal function defined $f_{\lambda,\mu,\alpha,\rho}(z)$ by (3.22).

Proof. Since $f(z) = z+\sum_{k=2}^{+\infty} a_k z^k \in B_\lambda(\mu,\alpha,\rho)$, we obtain

$$\begin{aligned}\operatorname{Re}&\left[(1-\mu)\left(\frac{D^\lambda f(z)}{z}\right)^{\alpha}+\mu(D^\lambda f(z))'\left(\frac{D^\lambda f(z)}{z}\right)^{\alpha-1}\right]\\ &= \operatorname{Re}\left[1+(1+\lambda)(\alpha+\mu)z+\cdots\right] > \rho.\end{aligned} \tag{3.33}$$

Therefore, it follows from Lemma 2.3 that

$$|(1+\lambda)(\alpha+\mu)a_2| \le 2-2\rho, \tag{3.34}$$

or

$$|a_2| \le \frac{2-2\rho}{(1+\lambda)(\alpha+\mu)}. \tag{3.35}$$

Note that $f_{\lambda,\mu,\alpha,\rho}(z) = z+((2-2\rho)/(1+\lambda)(\alpha+\mu))z^2+\cdots \in B_\lambda(\mu,\alpha,\rho)$, we obtain that inequality (3.32) is sharp. □

Remark 3.13. Setting $\lambda = 0$, $\mu = 1$ in Theorem 3.12, we get [8, Theorem 6].

Theorem 3.14 (covering theorem)**.** *Let $\alpha > 0$, $\mu > 0$, $\lambda \ge 0$, $\rho_0 \le \rho < 1$, $f(z) \in B_\lambda(\mu,\alpha,\rho)$, then the unit disk U is mapped on a domain that contain the disk $|w| < r_1$, where ρ_0 defined by (3.23) and*

$$r_1 = \frac{(1+\lambda)(\alpha+\mu)}{2-2\rho+2(1+\lambda)(\alpha+\mu)}. \tag{3.36}$$

Proof. Let w_0 be any complex number such that $f(z) \ne w_0$ for $z \in U$, then $w_0 \ne 0$ and

$$\frac{w_0 f(z)}{w_0 - f(z)} = z + \left(a_2 + \frac{1}{w_0}\right)z^2 + \cdots, \tag{3.37}$$

is univalent in U by Theorem 3.7, so

$$\left|a_2 + \frac{1}{w_0}\right| \le 2. \tag{3.38}$$

Therefore according to Theorem 3.12, we obtain

$$|w_0| \ge \frac{(1+\lambda)(\alpha+\mu)}{2-2\rho+2(1+\lambda)(\alpha+\mu)} = r_1. \tag{3.39}$$

Hence we have completed the proof. □

Setting $\lambda = 0$ and $\mu = 1$ in Theorem 3.14, we have the following corollary.

Corollary 3.15 (covering theorem)**.** *Let $f(z) \in B_0(1,\alpha,\rho)$ with $\mu > 0$, $0 \le \rho < 1$, then the unit disk U is mapped on a domain that contain the disk $|w| < (1+\alpha)/(4-2\rho+2\alpha)$.*

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